Strong Induction.

A poisoned cookie game

A square $n \times n$ of cookies.
The *top left is poisoned*.

Two players in turns can eat either:

- (the right) column or
- (the bottom) row of cookies.

You lose if you eat the poisoned cookie.

Does any of the players have a winning strategy?

A poisoned cookie game

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You lose if you eat the poisoned cookie.

Yes! *The second player has a winning strategy!* Can we prove it?

A poisoned cookie game

For the square 1×1, *Player One* eats the cookies and loses.

Inductive step. Assume *Player Two* has winning a strategy for a square $n \times n$.

Consider a bigger square, $(n+1) \times (n+1)$:

- Player One's move necessarily makes a rectangle $n \times (n-1)$. Then Player Two does the opposite move, reducing the rectangle to a square $n \times n$. And they have a winning strategy from now on.

A poisoned cookie game - 2

Change the rules:

What if the players can eat *multiple rows* or *multiple columns* of cookies?

A poisoned cookie game - 2

(A basically) the same strategy applies!

The inductive proof is a bit trickier:

- We *assume* that *for all* squares of *size* $\leq n$ a strategy exists.
- Prove that it exists *for all* squares of *size* $\leq (n+1)$.

  (Which can be done by just proving the case of the square $(n+1)\times(n+1)$, because all smaller squares have been covered by the assumption.)

This is called *Strong Induction*:
You assume that *all* previous cases work, and prove that the next one would as well!

A coin game

The game starts with a stack of n coins. In each move, you divide one stack into two nonempty stacks.

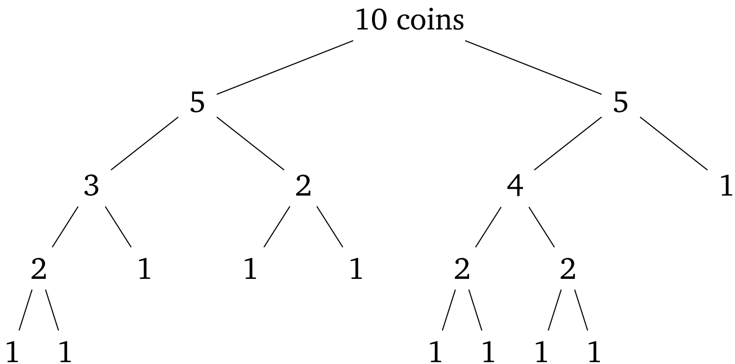
$$\begin{aligned} ||||| &\to ||| + || \\ &\to ||| + | + | \\ &\to || + | + | + | \\ &\to | + | + | + | + | \end{aligned}$$

If the new stacks have height a and b, then you score ab points for the move.

$$||||| \to ||| + || \qquad \text{you get } 3 \cdot 2 = 6 \text{ points}$$

What is the maximum score you can get?

A coin game



The total score: $25 + 6 + 4 + 2 + 1 + 4 + 1 + 1 + 1 = 45$ points.

Can we find a better strategy?

Unstacking n coins

Theorem. Every way of unstacking n coins gives a score of

$$S(n) = \frac{n(n-1)}{2} \text{ points.}$$

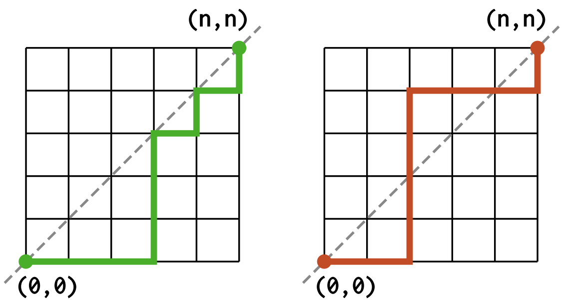
Proof by strong induction.
The base case: When $n = 1$, no moves is possible giving the score 0. The formula $1(1-1)/2 = 0$ works.

The inductive step: Assume any stack with $k \leq n$ coins gives $k(k+1)/2$ points. Prove that for any stack with $k \leq (n+1)$ the same formula applies.

$$\begin{aligned} S(n+1) &= S(k) + S(n+1-k) + k(n+1-k) \\ &= \frac{k(k-1)}{2} + \frac{(n+1-k)(n+1-k-1)}{2} + k(n+1-k) \\ &= \frac{1}{2}(k^2-k) + \frac{(n+1-k)(n-k)}{2} + kn + k - k^2 = \frac{(n+1)(n)}{2}. \end{aligned}$$

Count the number of good paths, C_n

Paths should go entirely below the diagonal line



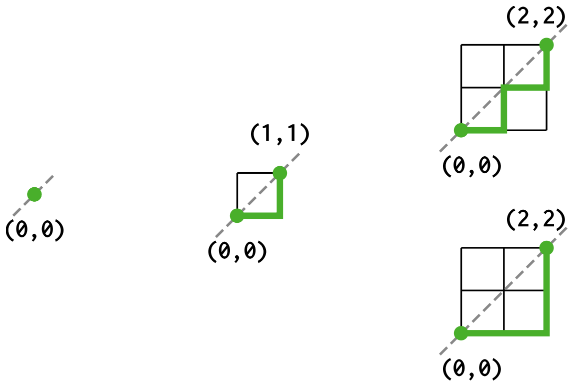
Good paths are below the diagonal

Bad paths cross it

The number of such paths, C_n, is the n^{th} Catalan number.

Count the number of good paths, C_n

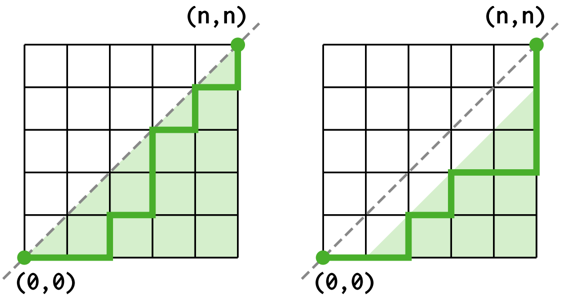
The cases when n is small: 0, 1, 2.



$C_0 = 1$ $C_1 = 1$ $C_2 = 2$

Recurrent formula

C_n is the number of paths that go below the diagonal (or touch the diagonal).



Introduce D_n, the number of paths that don't touch the diagonal in the middle points of the path.

Recurrent formula

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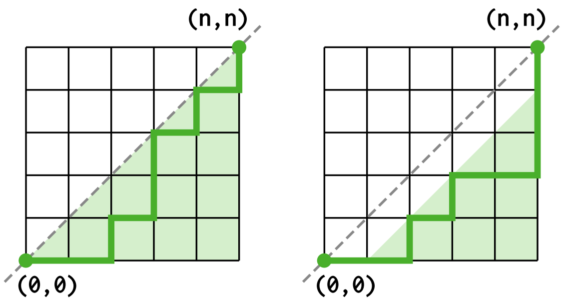
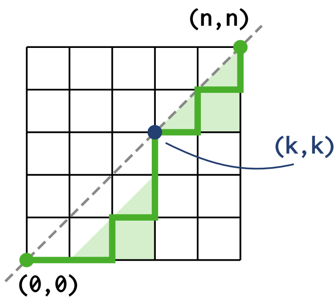


Introduce D_n, the number of paths that don't touch the diagonal in the middle points of the path.

$$D_n = C_{n-1}$$

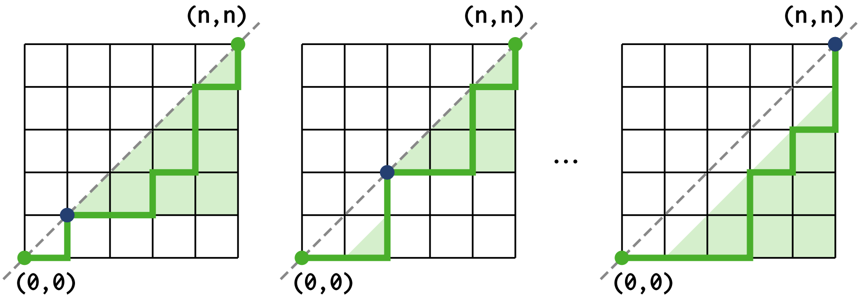
Recurrent formula

(k, k) be the first point of the given path that is on the diagonal and $k \neq 0$.



Given (k, k), the number of paths is $D_k C_{n-k}$

Recurrent formula

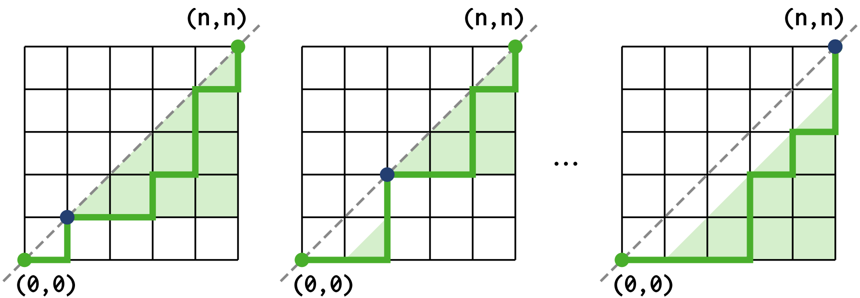


The diagonal point can be anywhere: $(1,1), (2,2), \ldots, (n,n)$

So, to count the total number of paths, we add up these n cases:

$$C_n = D_1 C_{n-1} + D_2 C_{n-2} + \ldots + D_n C_0 = \sum_{k=1}^{n} D_k C_{n-k}$$

Recurrent formula



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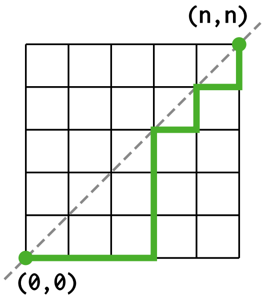
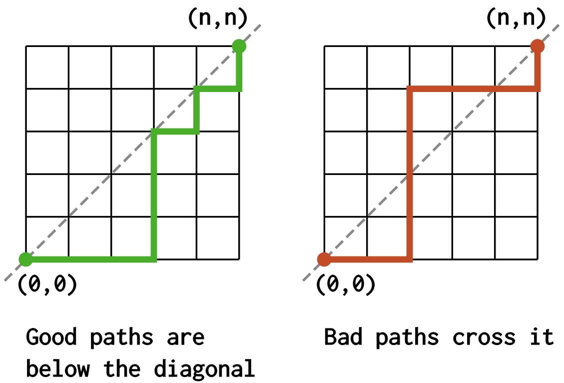
$$\text{since } D_k = C_{k-1}, \text{ we get } C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

Recurrent formula



$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

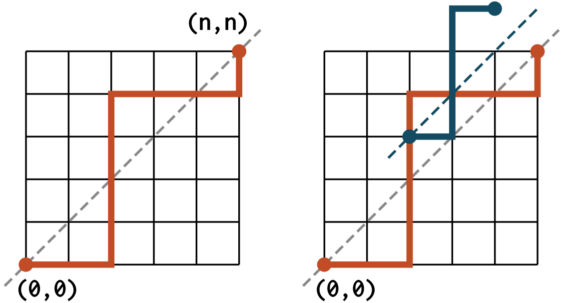
Closed form formula



We already know that the number of paths from the bottom-left to the top-right corner is $B_n = \binom{2n}{n}$

Let's try to count the number of paths that cross the diagonal, there is $B_n - C_n$ of them.

Closed form formula

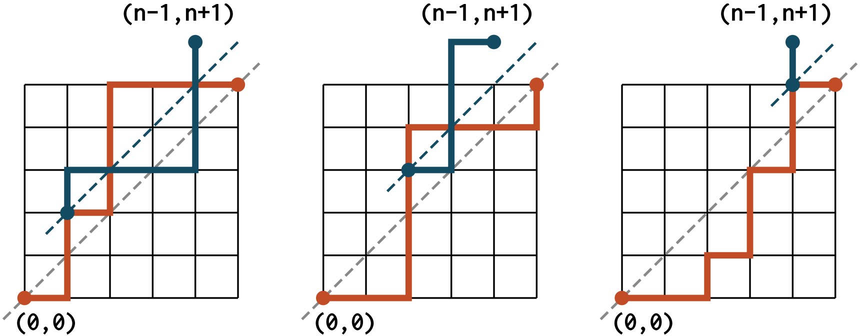


Consider a bad path that crosses the diagonal.

Lets say that the point $P = (k, k+1)$ is the first point above the diagonal. We mirror the remaining part of the path (shown in blue).

We can construct such new path for any invalid path.

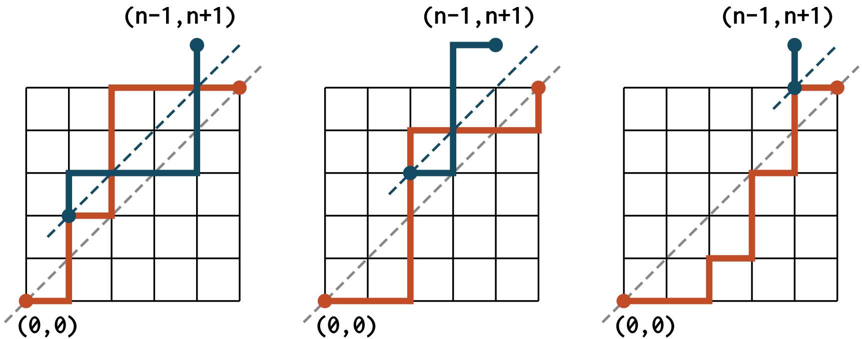
Closed form formula



Since we mirror the path starting at $P = (k, k+1)$, the remaining part of the path consisted of $(n-k, n-k-1)$ horizontal and vertical moves. Once reflected, it contains $(n-k-1, n-k)$ moves.

So, the resulting path ends up at the point
$Z = (k+n-k-1, k+1+n-k) = (n-1, n+1)$.
It does not depend on k.

Closed form formula

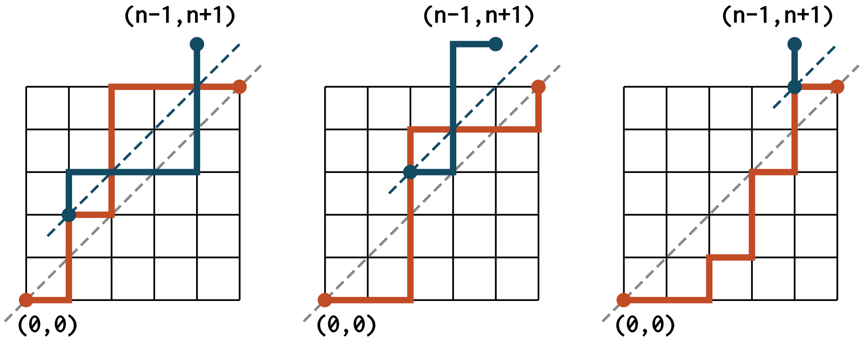


Every invalid paths becomes a path with $(n-1, n+1)$ horizontal and vertical moves.

So there is

$$B_n - C_n = \binom{n-1+n+1}{n+1} = \binom{2n}{n+1} \quad \text{of them.}$$

Closed form formula

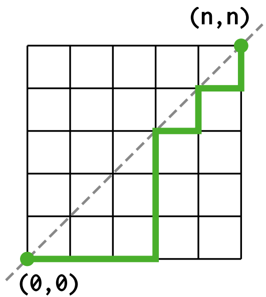


$$C_n = B_n - \binom{2n}{n+1}$$

Therefore,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{2} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

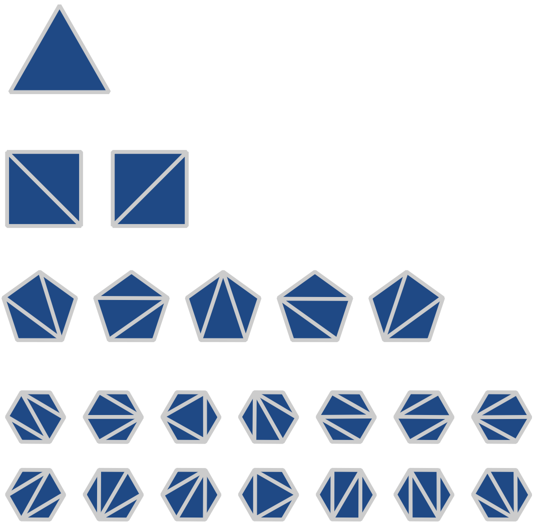
Three formulas for C_n



$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} \qquad C_n = \frac{1}{n+1}\binom{2n}{n}$$

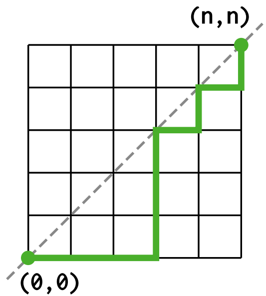
Catalan numbers are more than that



The number of ways to triangulate convex polygons:

1, 2, 5, 14, ...

Catalan numbers are more than that

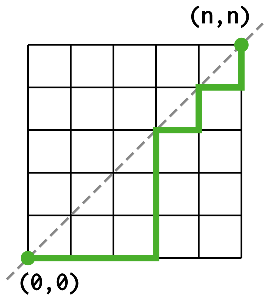


Let's encode the path with bits, $\{0, 1\}$.
If every move to the right is 1, and and every move up is 0:

$$1110001010$$

Well, not particularly interesting . . .

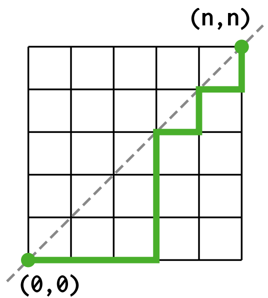
Catalan numbers are more than that



Let's encode the path with parentheses, {*(*,*)*}.
If every move to the right is *(*, and and every move up is *)*:

((( ))) ( ) ( )

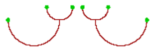
Catalan numbers are more than that



$((( ))) ( ) ( )$

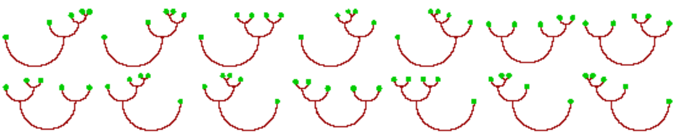
C_n is the number of strings made of n pairs of correctly balanced parentheses.

Catalan numbers are more than that



$C_2 = 2$

$C_3 = 5$



$C_4 = 14$

C_n is the number of full binary trees with $n + 1$ leaves:
$$2, 5, 14, \dots$$

(A rooted binary tree is full if every internal node has two children)