Discrete Structures. CSCI-150. Spring 2017.

Homework 5. **Due Mon. Mar 6, 2017.**

Problem 1 (Graded)

In how many ways can 21 identical computers be distributed among 5 computer stores if

(a) there are no restrictions?

(b) each store gets at least two?

(c) the largest store gets no less than half?

(d) each store gets at least four?

Problem 2

Find the number of integer solutions to the equation

$$w + x + y + z = 19,$$

where the variables are positive integers.

Pigeonhole principle

Problem 3 (Graded)

(a) There are 50 white socks and 50 black socks in a drawer. How many socks do you have to take to be sure that you have at least one matching pair?

(b) At least one mismatching pair?

Problem 4

Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than 1/2.

Problem 5 (Graded)

In a room there are 10 people, none of them is younger than 1 or older than 100 years. Prove that among them, one can always find two groups of people (possibly intersecting, but different) the sums of whose ages are the same.

Hint: Use the pigeonhole principle. How many different groups of people can be picked? How large the sum of their ages can be?

Hint 2: Each group $\rightarrow$ sum of ages. You want to show that at least two groups will $\rightarrow$ the same sum of ages.

Proving identities and Double counting

Problem 6 (Graded)

In this problem, you have to find two proofs for the following identity:

$$n \cdot \binom{n-1}{k-1} = \binom{n}{k} \cdot k$$

(a) The first proof is algebraic. Using the fact that $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$, show that the equation is always true. It may involve some factorial manipulation, but almost everything should cancel out. You may consider the left-hand side and the right-hand side separately, showing that both simplify to the same formula.

Write all simplification steps as a *contiguous chain of equal formulas*, it should look like

$$\binom{n+1}{n} = \frac{(n+1)!}{(n+1-n)! \cdot n!} = \frac{n! \cdot (n+1)}{1! \cdot n!} = \frac{n+1}{1} = n+1.$$

So it is *not ambiguous* where each formula is coming from.

(b) For the second part, prove the same identity using the technique called "Double counting" or "Combinatorial argument". It's a combinatorial proof technique for showing that two expressions are equal by demonstrating that they are two ways of counting the size of one set.

In other words, we need to come up with a "story" for what both, the left-hand side and the right-hand side, are counting.

Problem 7

Find "double counting" proofs for the following identities:

$$\binom{n-1+2}{2} = n + \binom{n}{2}$$

$$\binom{n-1+3}{3} = n + \binom{n}{2} \cdot 2 + \binom{n}{3}$$

$$\binom{n+m}{2} = \binom{n}{2} + \binom{m}{2} + nm$$

$$(2n)! = \binom{2n}{n} \cdot (n!)^2$$

Hint: when proving the first two identities, think of the selection with repetition.