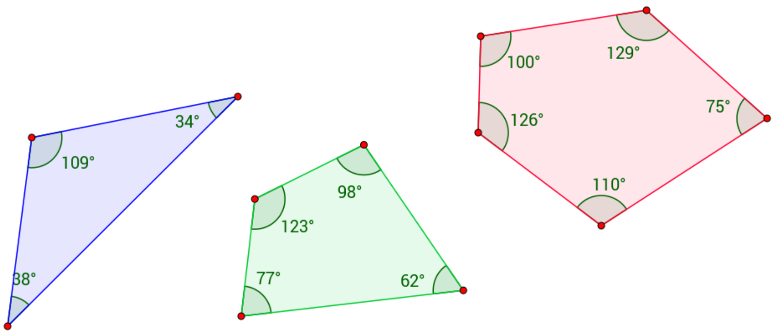
Induction

Example. Consider a problem

Show that the **sum of all interior angles** of a simple poligon with n vertices (you may assime it is convex) is equal to $180°(n-2)$.



Example 2. Consider a problem

Prove that

$$n^3 + 2n$$

is a multiple of 3 for all positive integers $n = 1, 2, 3, \ldots$.

$$3, \quad 12, \quad 33, \quad 72, \quad 135, \quad \ldots$$

Another problem

Let's prove that the sum of the first n positive integers

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

for all positive integers $n = 1, 2, 3, 4, \ldots$

Another problem

Let

$$P(n):\ 1+2+\ldots+n \ = \ \frac{n(n+1)}{2}.$$

Prove that

$$P(n)$$

for all positive natural numbers $n = 1,\, 2,\, 3,\, 4 \ldots$

The idea

If we can prove that

$$
\begin{aligned}
&P(1) \\
&P(1) \to P(2) \\
&P(2) \to P(3) \\
&\ldots \\
&P(n) \to P(n+1) \\
&\ldots
\end{aligned}
$$

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

The idea

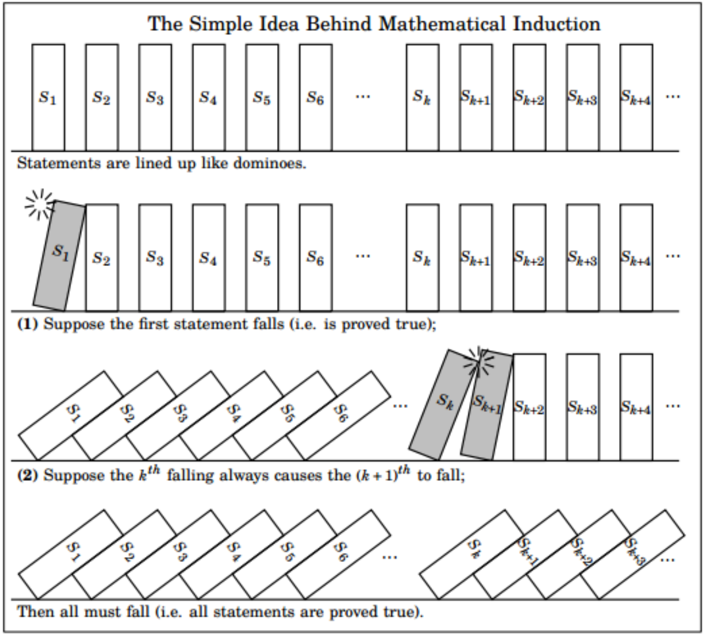
The implications can be grouped together. Thus it is enough to prove that

$$1)\quad P(1)$$
$$2)\quad \forall n \geq 1 \;:\; P(n) \rightarrow P(n+1)$$

Then it follows that

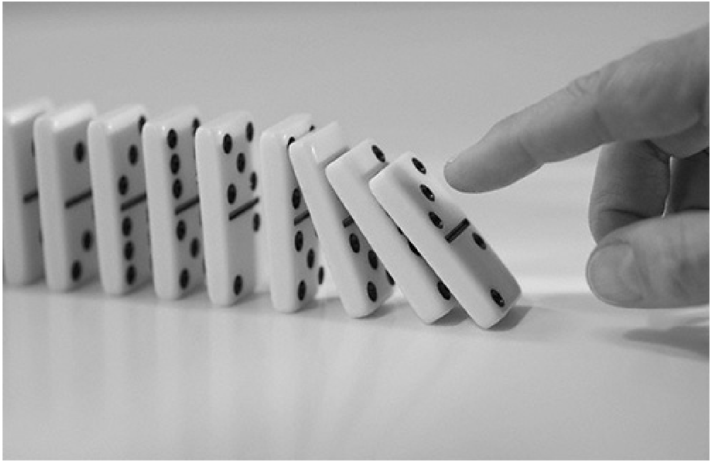
$$P(n) \text{ for all } n \geq 1$$

Principle of induction



Principle of induction

Principle of induction



Principle of induction

If we can prove

1. *“The basis step”*

$$P(1) \text{ is true, and}$$

2. *“The inductive step”*

$$\text{for all } n \geq 1, \;\; P(n) \quad \text{implies} \quad P(n+1).$$

then $P(n)$ is true for every integer $n \geq 1$.

Example proof

Let's use induction to prove the formula of the sum of positive integers from 1 to n:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(We have to show that both, the *basis step* and the *inductive step* are correct)

Example proof

Prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Part 1. *The basis step.* Consider $n = 1$.

The left hand side is just

$$1$$

The right-hand side:

$$\frac{1 \cdot (1+1)}{2} = 1$$

They are equal, so it is true.

Example proof

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n = m \geq 1$:

$$1 + 2 + 3 + \ldots + m = \frac{m(m+1)}{2}$$

We have to prove that it is also true for the next value $n = m + 1$:

$$1 + 2 + 3 + \ldots + m + (m+1) = \frac{(m+1)((m+1)+1)}{2}$$

Example proof

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n = m \geq 1$:

$$1 + 2 + 3 + \ldots + m = \frac{m(m+1)}{2}$$

We have to prove that it is also true for the next value $n = m + 1$:

$$1 + 2 + 3 + \ldots + m + (m+1) = \frac{(m+1)((m+1)+1)}{2}$$

Consider the left-hand side:

$$\underbrace{1 + 2 + 3 + \ldots + m}_{= \frac{m(m+1)}{2}} + (m+1) = \frac{m(m+1)}{2} + m + 1 = \frac{m(m+1) + 2(m+1)}{2}$$

$$= \frac{(m+1)(m+2)}{2} = \frac{(m+1)((m+1)+1)}{2}$$

Example proof

The base case and the inductive step are true.

Therefore, *by induction*, the formula is correct for every positive integer n.

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

This is a very useful formula, by the way.

Rewrite it using the sigma-notation

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

Prove by induction

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for all } n \geq k$$

Prove by induction

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for all } n \geq k$$

That is, for all $n \geq k$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Example proof 2

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for all } n \geq k$$

Choose some integer k. We have to show that then, for all $n \geq k$, the identity holds.

The base case. $n = k$.

The left hand side:

$$\sum_{i=k}^{k} \binom{i}{k} = \binom{k}{k} = 1.$$

The right hand side: $\binom{k+1}{k+1} = 1$. Both are equal.

Example proof 2

The inductive step.

Assume that the equality holds for some $n \geq k$:

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Prove that it also holds for $n+1$:

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{(n+1)+1}{k+1}$$

To prove that, take the left hand side, and show that it is equal to the right hand side.

Example proof 2

Prove that

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{(n+1)+1}{k+1}$$

The left hand side:

$$\sum_{i=k}^{n+1} \binom{i}{k} = \sum_{i=k}^{n} \binom{i}{k} + \binom{n+1}{k}$$

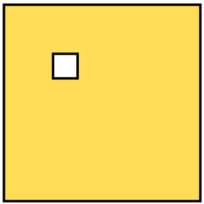
By the inductive hypothesis, the sum $\sum_{i=k}^{n} = \binom{n+1}{k+1}$, so

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1},$$

where the last equality holds because of Pascal's identuty.

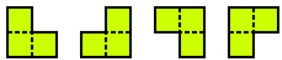
Therefore, the statement is true by induction.

Tiling $2^n \times 2^n$ with 1 square removed

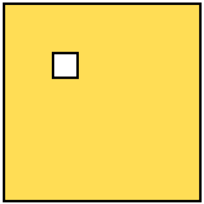


For all n>0, a checkerboard 2^n x 2^n with one square removed

can be tiled by L-shaped tiles

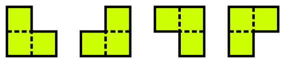


Tiling $2^n \times 2^n$ with 1 square removed

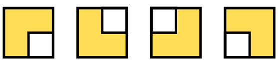


For all n>0, a checkerboard 2^n x 2^n with one square removed

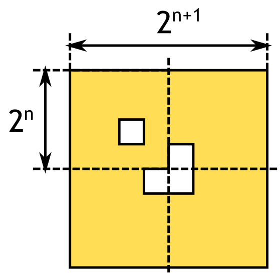
can be tiled by L-shaped tiles



The base case for n=1



Inductive step.
Assuming that we can tile 2^n x 2^n with one removed, prove that it's possible to tile 2^{n+1} x 2^{n+1} with one removed



Another example proof

Let $P(n)$ be the predicate, "*I can lift n grains of sand*."

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.

Another example proof

Let $P(n)$ be the predicate, "I can lift n grains of sand."

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.
- Then, surely, if I can lift m grains, then I can lift $m+1$, it does not make any difference!

$$P(m) \rightarrow P(m+1)$$

  This is my inductive step.

Another example proof

Let $P(n)$ be the predicate, "I can lift n grains of sand."

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.
- Then, surely, if I can lift m grains, then I can lift $m+1$, it does not make any difference!

$$P(m) \rightarrow P(m+1)$$

  This is my inductive step.

So, by induction, *I can lift any amount of sand*. Right?

Where is a mistake?

Of course, the proof is wrong. But should we blame induction for that?

Well, we made an error in the proof of $P(m) \to P(m+1)$.

It is hard to say for exactly which m it is false, but certainly there is some value!

Sum of b^k

Prove by induction that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

First we can do the base case, then the inductive step.

Sum of b^k

Prove that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Basis step ($n = 0$):

$$b^0 = 1, \quad \text{and} \quad \frac{b^1 - 1}{b - 1} = 1$$

.

Sum of b^k

Prove that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Inductive step:

As always, we make a hypothesis that for some $n = m \geq 0$ the formula holds:

$$b^0 + b^1 + b^2 + \ldots + b^m = \frac{b^{m+1} - 1}{b - 1}$$

And we have to prove that the formula is correct for $n = m + 1$:

$$b^0 + b^1 + b^2 + \ldots + b^m + b^{m+1} = \frac{b^{m+2} - 1}{b - 1}$$

Sum of b^k

Inductive step:

We have to prove that the formula is correct for $n = m + 1$:

$$b^0 + b^1 + b^2 + \ldots + b^m + b^{m+1} = \frac{b^{m+2} - 1}{b - 1}$$

$$\underbrace{b^0 + b^1 + b^2 + \ldots + b^m}_{=\frac{b^{m+1}-1}{b-1} \text{ by the hypothesis}} + b^{m+1} = \frac{b^{m+1} - 1}{b - 1} + b^{m+1}$$

$$= \frac{b^{m+1} - 1 + b^{m+2} - b^{m+1}}{b - 1} = \frac{b^{m+2} - 1}{b - 1}.$$

Sum of b^k

So, this formula for the sum is correct

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

In the sigma-notation:

$$\sum_{k=0}^{n} b^k = \frac{b^{n+1} - 1}{b - 1}$$

Sum of b^k

We can multiply both sides by a constant a:

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1},$$

The sequence of numbers

$$a,\ ab,\ ab^2,\ ab^3,\ \ldots\ ab^n,\ \ldots$$

is called a *Geometric progression*.

So, we proved the formula for the partial sum of a geometic progression.

Sum of kb^{k-1}

The partial sum of the geometric progression with the coefficient $a = 1$:

$$\sum_{k=0}^{n} b^k = b^0 + b^1 + b^2 + \ldots + b^n = \frac{(b^{n+1} - 1)}{b - 1},$$

Can we compute

$$\sum_{k=0}^{n} kb^{k-1} = 0 + 1 + 2b + 3b^2 + 4b^3 \ldots + nb^{n-1} \quad ?$$

The terms b^k have these increasing coefficients now ...

Sum of kb^{k-1}

There is a cheap trick. We know that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{(b^{n+1} - 1)}{b - 1}$$

If two functions of the same argument are equal, then their derivatives with respect to that argument are equal too

$$\frac{d}{db}\left(b^0 + b^1 + b^2 + \ldots + b^n\right) = \frac{d}{db}\left(\frac{(b^{n+1} - 1)}{b - 1}\right)$$

$$0 + 1 + 2b + 3b^2 + 4b^3 + \ldots + nb^{n-1} = \frac{d}{db}\left(\frac{b^{n+1} - 1}{b - 1}\right) = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}$$

$$\text{Therefore, } \sum_{k=0}^{n} kb^{k-1} = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}.$$

Geometric progression again

The partial sum of the geometric progression:

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1}-1)}{b-1}$$

Well, what if we want to add up an infinite sequence?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Infinite geometric progression

The partial sum of the geometric progression is

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1}-1)}{b-1}$$

If b is a small real number, specifically, if the absolute value $|b| < 1$, then

$$|b| > |b^2| > |b^3| > \ldots$$

In the limit, $\lim_{n\to\infty} b^n = 0$

$$\sum_{n=0}^{\infty} ab^n = \lim_{n\to\infty} ab^n = \lim_{n\to\infty} \frac{a(b^{n+1}-1)}{b-1} = \frac{a(-1)}{b-1} = \frac{a}{1-b}$$

Proving inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

Proving inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

The basis step:
$n = 1$. $2 > 1$ is true.

Proving inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

The basis step:
$n = 1$. $2 > 1$ is true.

The inductive step:
Assume that $2^n > n$ for $n \geq 1$. Prove that $2^{n+1} > (n+1)$.

Equivalently, we have to prove that

$$2^{n+1} - (n+1) > 0.$$

Proving inequalities

We assumed that $2^n > n$ for $n \geq 1$.

We have to prove that $2^{n+1} - (n+1) > 0$.

$$
\begin{aligned}
2^{n+1} - (n+1) &= 2 \cdot 2^n - n - 1 \\
&> 2 \cdot n - n - 1 \qquad \text{(by the I.H.)} \\
&= n - 1 \\
&\geq 1 - 1 = 0. \qquad (\text{b/c } n \geq 1)
\end{aligned}
$$

Therefore, *by induction*, $2^n > n$ is true for $n \geq 1$.

One more proof

Theorem. All horses are the same color.

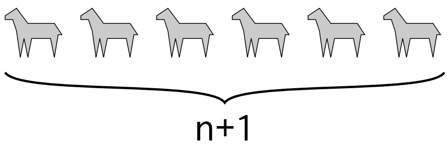
We can prove this by induction on the number of horses in a given set.

All horses are the same color

The basis step. If there's just one horse then it's the same color as itself.

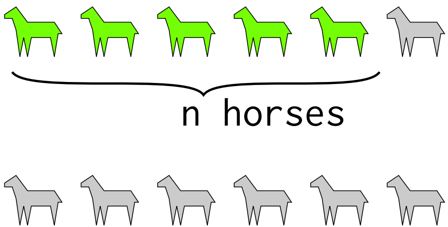
For the *inductive step*, assume that n horses are of the same color.

Assume that there are $n+1$ horses numbered 1 to $n+1$.



All horses are the same color

By the induction hypothesis, horses 1 through n are the same color, and similarly horses 2 through $n+1$ are the same color.

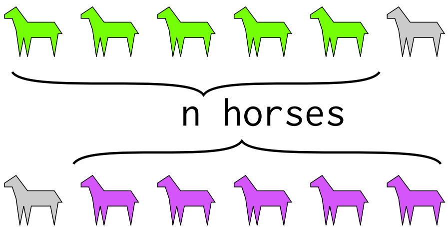


But the middle horses, 2 through n , can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and $n+1$ must be the same color as well. Thus all $n+1$ horses are the same color.

What, if anything, is wrong with this reasoning?

All horses are the same color

By the induction hypothesis, horses 1 through n are the same color, and similarly horses 2 through $n+1$ are the same color.



But the middle horses, 2 through n , can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and $n+1$ must be the same color as well. Thus all $n+1$ horses are the same color.

What, if anything, is wrong with this reasoning?