Binomial Theorem.
Combinations with repetition.

Permutations and combinations

Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* ("n choose r"):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\, r!}$$

What are the properties of $\binom{n}{r}$?

How does $\binom{n}{r}$ change with r?

$$\binom{0}{0} = \frac{0!}{0!\,0!} = 1.$$

$$\binom{1}{0} = \frac{1!}{1!\,0!} = 1, \quad \binom{1}{1} = \frac{1!}{0!\,1!} = 1.$$

$$\binom{2}{0} = \frac{2!}{2!\,0!} = 1, \quad \binom{2}{1} = \frac{2!}{1!\,1!} = 2, \quad \binom{2}{2} = \frac{2!}{0!\,2!} = 1.$$

$$\binom{3}{0} = \frac{3!}{3!\,0!} = 1, \quad \binom{3}{1} = \frac{3!}{2!\,1!} = 3, \quad \binom{3}{2} = \frac{3!}{1!\,2!} = 3, \quad \binom{3}{3} = \frac{3!}{0!\,3!} = 1.$$

Pascal’s Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^0 = 1$$

$$
\begin{aligned}
(x+y)^1 &= x + y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

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$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

Example:

$$\begin{aligned}(x+y)^3 =& (x+y)(x+y)(x+y) = \\ & xxx + \\ & xxy + xyx + yxx + \\ & xyy + yxy + yyx + \\ & yyy\end{aligned}$$

$2^n = 2^3 = 8$ terms in total. The same as the number of the bit strings of length 3.

Coefficients of $(x + y)^n$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} = \quad ?$$

What is happening when we multiply $(x + y)$ n times?

Coefficients of $(x+y)^n$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} = \quad ?$$

What is happening when we multiply $(x+y)$ n times?

We get the sum of

$$\begin{aligned}
&xxxxx\ldots x+\\
&yxxxx\ldots x+\\
&xyxxx\ldots x+\\
&yyxxx\ldots x+\\
&xxyxx\ldots x+\\
&\ldots+\\
&yyyyy\ldots y
\end{aligned}$$

Coefficients of $(x+y)^n$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$\underbrace{x\cdot x\cdot\ldots\cdot x}_{=x^n} +$$

$$(\underbrace{x\cdot\ldots\cdot x}_{=x^{n-1}})\cdot y + \ldots + y\cdot(x\cdot\ldots\cdot x) +$$

$$(\underbrace{x\cdot\ldots\cdot x}_{=x^{n-2}})\cdot(y\cdot y) + \ldots + (y\cdot y)\cdot(\underbrace{x\cdot\ldots\cdot x}_{=x^{n-2}}) +$$

$$\ldots +$$

$$\underbrace{y\cdot y\cdot\ldots\cdot y}_{=y^n}$$

Coefficients of $(x+y)^n$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$1\cdot x^n +$$

$$n\cdot x^{n-1}y +$$

$$\binom{n}{2}\cdot x^{n-2}y^2 +$$

$$\ldots +$$

$$1\cdot y^n$$

The Binomial Theorem

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$\binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

Shorter notation for the same thing:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

This result is called *The Binomial Theorem*, and this is why the coefficients, $\binom{n}{k}$, are also called the *binomial coefficients*.

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

The Binomial Theorem

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$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 = \sum_{k=0}^{n} \binom{n}{k}.$$

Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The Binomial Theorem

$$\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=2^n.$$

Results like this are not very obvious.

Recall that

$$\binom{n}{k}=\frac{n!}{(n-k)!\,k!}$$

So,

$$\frac{n!}{n!\,0!}+\frac{n!}{(n-1)!\,1!}+\frac{n!}{(n-2)!\,2!}+\ldots+\frac{n!}{0!\,n!}=2^n.$$

In this form, the result seems to be much harder to prove.

However, can you think of another way to prove this identity?
(You can try to use double counting)

The Binomial Theorem

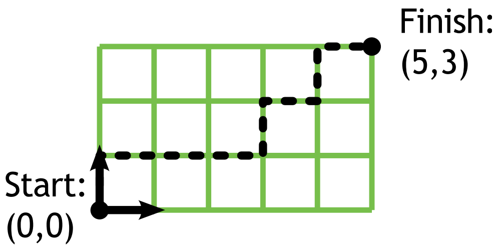
Using the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

prove that

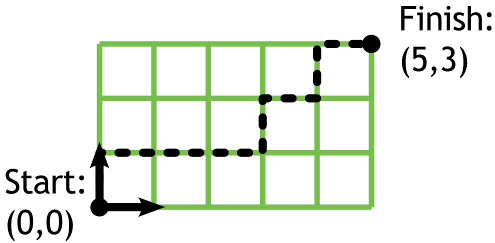
$$\sum_{k=0}^{n} 2^k \binom{n}{k} = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} = 3^n.$$

Counting routes



You can go only North and East.
Count the number of paths from $(0, 0)$ to $(5, 3)$.

Counting routes



You can go only North and East.
Count the number of paths from $(0,0)$ to $(5,3)$. Answer: $\binom{5+3}{3}$.

Pascal's Triangle Again

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Every number in Pascal's triangle is equal to the sum of the two numbers that are immediately above.

Another Identity

Prove that for $r \leq n$ and $r \leq m$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Vandermonde's Identity

Prove that for $r \leq n$ and $r \leq m$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

We split the initial set of $m+n$ objsects into two arbitrary subsets of m and n objects. After that, we can choose r objects from the two subsets in the following ways:

k	subset of size m	subset of size n
0	choose r	choose none
1	choose $r-1$	choose 1
2	choose $r-2$	choose 2
…		
r	choose 0	choose r

$$\binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \ldots + \binom{m}{0}\binom{n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

r-combinations without repetition

Recall that without repetitions, this is $\binom{n}{r}$.

For example, you have n books, but don't have time to read all of them, and have to select only r books to read.

In how many ways can you do so?

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

There are $\binom{n}{r}$ ways to make the choice.

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\$ \quad \$ \quad \$ \quad \$ \quad \$$$

r-combinations with repetition

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$$\$ \mid \$ \quad \$ \mid \$ \quad \$$$

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\underbrace{\quad \$ \quad}_{\text{1 drink}} \mid \underbrace{\$ \quad \$}_{\text{2 drinks}} \mid \underbrace{\$ \quad \$}_{\text{2 drinks}}$$

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$$\$ \quad \$ \mid \$ \quad \$ \quad \$ \mid$$

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\underbrace{\$ \quad \$}_{\text{2 drink}} \mid \underbrace{\$ \quad \$ \quad \$}_{\text{3 drinks}} \mid \underbrace{\qquad}_{\text{none}}$$

r-combinations with repetition

So, there are $5+(3-1)$ places that stand for 5 dollars and $(3-1)$ separators between the drinks' types.

$$\_\;\_\;\_\;\_\;\_\;\_\;\_$$

$$\$ \mid \$\ \$ \mid \$\ \$$$

$$\$\ \$ \mid \$\ \$\ \$ \mid$$

$$\$\ \$\ \$\ \$\ \$ \mid \mid$$

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$$\_\ \_\ \_\ \_\ \_\ \_\ \_$$

$$\$ \mid \$\ \$ \mid \$\ \$$$

$$\$\ \$ \mid \$\ \$\ \$ \mid$$

$$\$\ \$\ \$\ \$\ \$ \mid \mid$$

$$\binom{7}{5} = \binom{7}{2} = 21 \text{ ways to buy 5 drinks}$$

r-combinations with repetition

To select r objects out of n with repetitions, there are

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1} \text{ ways.}$$

_ _ _ _ _ _ _ _ _ _

* * * * * * | | | |

(r objects and $n-1$ separator)

In other words, this is the number of r-combinations with repetition from the set of n objects.

r-permutations with repetition

We know that the number of r-permutations of n objects *without repetition* is

$$n(n-1)(n-2)\cdot\ldots\cdot(n-r+1) = \frac{n!}{(n-r)!}$$

But, *if repetitions are allowed*, it is even easier, the simple product rule works just fine!

$$n \cdot n \cdot \ldots \cdot n = n^r$$

Summary

	with repetitions?	
r-combination	No	$\binom{n}{r}$
r-combination	Yes	$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$
r-permutation	No	$P(n,r) = \frac{n!}{(n-r)!}$
r-permutation	Yes	n^r