

# Counting

# What can we count?

- In how many ways can we paint 6 rooms, choosing from 15 available colors?
- What if we want all rooms painted with different colors?
- In how many different ways 10 books can be arranged on a shelf?
- What if 2 of those 10 books are identical copies?

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

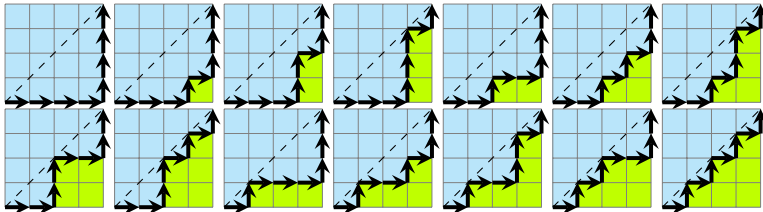
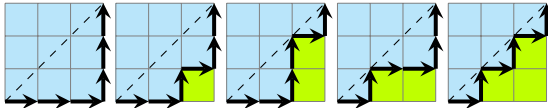
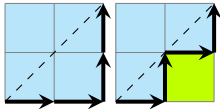
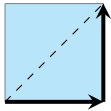
Subtraction Rule

Counting

Tree Diagrams

Factorial

# 1, 2, 5, 14, ...



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1, 2, 5, 14, ...



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# Problem

The chairs of an auditorium are to be labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1, A-2, ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

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# Problem

The chairs of an auditorium are to be labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1, A-2, ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

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# The Rule of Product

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There are

1. 26 ways to assign a letter and

$$26 \cdot 100$$

2. 100 ways to assign a number.

*The Rule of Product.* Suppose that a procedure can be broken down into a sequence of two tasks. If there are  $n_1$  ways to do the first task and for each of these ways of doing the first task, there are  $n_2$  ways to do the second task, then there are  $n_1 n_2$  ways to do the procedure.

# Chairs again

Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

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# Chairs again

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Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

```
for  $a := A$  to  $Z$  do  
  for  $n := 1$  to 100 do  
    print_label(a, n)
```

# Generalized Product Rule

If a procedure consists of  $k$  *sub-tasks*, and the sub-tasks can be performed in  $n_1, \dots, n_k$  ways, then the procedure can be performed in  $(n_1 \cdot n_2 \cdot \dots \cdot n_k)$  ways.

Example:

Count the number of different bit strings of length seven.

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# Generalized Product Rule

If a procedure consists of  $k$  *sub-tasks*, and the sub-tasks can be performed in  $n_1, \dots, n_k$  ways, then the procedure can be performed in  $(n_1 \cdot n_2 \cdot \dots \cdot n_k)$  ways.

Example:

Count the number of different bit strings of length seven.

The value for each bit can be chosen in two ways (0 or 1). Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

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# License plates



How many different license plates of this format can be made?

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# License plates



How many different license plates of this format can be made?

$$26^3 \cdot 10^4 = 175,760,000$$

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# Another counting problem

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A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

# Another counting problem

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

$40 + 50 = 90$  this is called the rule of sum

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# The Rule of Sum

40 books on sociology, and 50 books on anthropology.  
There are  $40 + 50 = 90$  ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

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# The Rule of Sum

40 books on sociology, and 50 books on anthropology.  
There are  $40 + 50 = 90$  ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

```
for  $b := 1$  to 40 do  
    read(Sociology,  $b$ )  
for  $b := 1$  to 50 do  
    read(Anthropology,  $b$ )
```

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# The Rule of Sum

40 books on sociology, and 50 books on anthropology.  
There are  $40 + 50 = 90$  ways to choose a book.

*The Rule of Sum.* If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways, where none of the set of  $n_1$  ways is the same as any of the set of  $n_2$  ways, then there are  $n_1 + n_2$  ways to do the task.

Note that it's important that the two groups don't have common elements (We say that they are disjoint sets).

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# Problem

You have 3 textbooks, 5 novels, 4 magazines, and 2 comic books.

You want to pick only one book to read in the subway. How many options do you have?

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# Problem

You have 3 textbooks, 5 novels, 4 magazines, and 2 comic books.

You want to pick only one book to read in the subway. How many options do you have?

$$3 + 5 + 4 + 2 = 14.$$

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# Problem

NYS wants to change the license plates format, allowing  
3 letters + 3 digits; 2 letters + 2 digits; and 1 letter + 1 digit.

AAA 111

AA 11

A 1

How many license plates can be made?

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# Problem

NYS wants to change the license plates format, allowing up to 3 letters followed by up to 3 digits.

A 1	A 11	A 111
AA 1	AA 11	AA 111
AAA 1	AAA 11	AAA 111

How many license plates can be made?

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# A new object

**Def.** A *set* is an unordered collection of objects. The objects are called elements.

If  $e$  is an element of the set  $A$ , we write  $e \in A$ .

Otherwise, if it's not in  $A$ , we write  $e \notin A$ .

Example:

$$A = \{1, 2, 97, 3, 15\}.$$

$$1 \in A.$$

$$4 \notin A.$$

$$\{1, 2, 3, 3, 2, 1, 1, 1, 1\} = \{1, 2, 3\}$$

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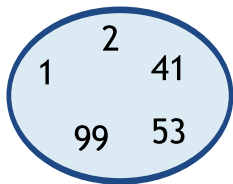
Subtraction Rule

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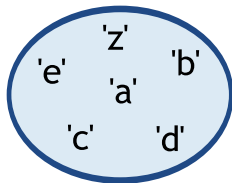
Tree Diagrams

Factorial

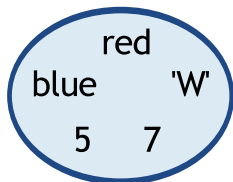
# Sets



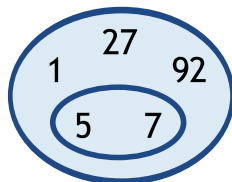
$$A = \{1, 2, 41, 53, 99\}$$



$$B = \{'a', 'z', 'e', 'd', 'c', 'b'\}$$



$$C = \{'W', \text{blue}, 5, \text{red}, 7\}$$



$$D = \{27, 1, \{5, 7\}, 92\}$$

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# Some important sets

*Natural numbers*

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

*Integer numbers*

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

*Empty set*

$$\emptyset = \{ \}$$

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# Set Builder Notation

We can describe sets using predicates:

$$A = \{x \mid P(x)\}$$

“Set  $A$  is such that  $x \in A$  if and only if  $P(x)$ .”

Example. Positive integers:

$$\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1, 2, 3, \dots\}$$

More complex predicates are fine too. Odd and even numbers:

$$\text{Even} = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (n = 2k)\}$$

$$\text{Odd} = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (n = 2k + 1)\}$$

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# Union, $\cup$

$A \cup B$  denotes all things that are *members of either A or B*:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Equivalently:

$x$  belongs to  $A \cup B$  if and only if  $x \in A$  or  $x \in B$ .

Examples:

$$\begin{aligned}\{1, 2\} \cup \{‘a’, ‘b’\} &= \{1, 2, ‘a’, ‘b’\} \\ \{1, 2, 3\} \cup \{2, 3, 5\} &= \{1, 2, 3, 5\}\end{aligned}$$

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# Intersection, $\cap$

$A \cap B$  denotes all things that are *members of both A and B*:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Equivalently:

$x$  belongs to  $A \cap B$  if and only if  $x \in A$  and  $x \in B$ .

Examples:

$$\{1, 2\} \cap \{\text{'a'}, \text{'b'}\} = \emptyset$$

$$\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$$

Sets  $A$  and  $B$  are called *disjoint* if their intersection is empty:

$$A \cap B = \emptyset.$$

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# Number of the elements of a finite set

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**Def.** If set  $A$  is finite, and there are exactly  $n$  elements in  $S$ , then  $n$  is the *cardinality* of the set  $A$ . We write

$$|A| = n.$$

Examples:

$$A = \{3, 4, 5, 6\}$$

$$|A| = 4$$

$$B = \{\{3, 4\}, \{5, 6\}, 7\}$$

$$|B| = 3$$

$$|\emptyset| = 0$$

# Question

$$A = \{1, 2, 4, 5\}$$

$$B = \{20, 21, 22, 23, 24\}$$

$$A \cup B = \{1, 2, 4, 5, 20, 21, 22, 23, 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality of their union?

$$|A \cup B| =$$

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# Question

The Rule of Product

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$$B = \{20, 21, 22, 23, 24\}$$

$$A \cup B = \{1, 2, 4, 5, 20, 21, 22, 23, 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| = 9, \quad \text{and} \quad |A| + |B| = 4 + 5 = 9.$$

$$|A \cup B| = |A| + |B| = 4 + 5 = 9.$$

# Question

You are given  $k$  disjoint sets  $A_1, \dots, A_k$ :

It means that  $A_i \cap A_j = \emptyset$  when  $i \neq j$ .

What is the cardinality of their union  $A_1 \cup \dots \cup A_k$ ?

$$|A_1 \cup \dots \cup A_k| =$$

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What is the cardinality of their union  $A_1 \cup \dots \cup A_k$ ?

$$|A_1 \cup \dots \cup A_k| = |A_1| + \dots + |A_k| = \sum_{i=1}^k |A_i|.$$

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# Question

The Rule of Product

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$$|A_1 \cup \dots \cup A_k| = |A_1| + \dots + |A_k| = \sum_{i=1}^k |A_i|.$$

This is the same *rule of sum*, right?

# Question

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Why do we insist on the sets being disjoint?

Really, who cares?

# Because

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

Their union:  $A \cup B = \{1, 2, 3, 4\}$

$$|A| + |B| = |\{1, 2, 3\}| + |\{3, 4\}| = 3 + 2 = 5$$

$$|A \cup B| = |\{1, 2, 3, 4\}| = 4$$

So, in general,  $|A \cup B| \neq |A| + |B|$ , and if we try to use the sum rule when the sets are not disjoint, we *overcount*, and this is really bad.

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# Overcounting

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

$$A \cup B = \{1, 2, 3, 4\}$$

We were overcounting, because the common elements of  $A$  and  $B$  were counted twice:

$$|A| + |B| = |\{1, 2, 3\}| + |\{3, 4\}| = 5,$$

$$|A \cup B| = |\{1, 2, 3, 4\}| = 4.$$

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Therefore, to get the correct value of  $|A \cup B|$ , the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - |\{3\}| = 3 + 2 - 1 = 4.$$

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# The Subtraction Rule

$$A = \{1, 2, 3\}$$

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Therefore, to get the correct value of  $|A \cup B|$ , the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - |\{3\}| = 3 + 2 - 1 = 4.$$

The *Subtraction Rule* for two arbitrary sets  $A$  and  $B$ :

$$|A \cup B| =$$

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# The Subtraction Rule

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Therefore, to get the correct value of  $|A \cup B|$ , the number of common elements must be subtracted:

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The *Subtraction Rule* for two arbitrary sets  $A$  and  $B$ :

$$|A \cup B| = |A| + |B| - |A \cap B|$$

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**Subtraction Rule**

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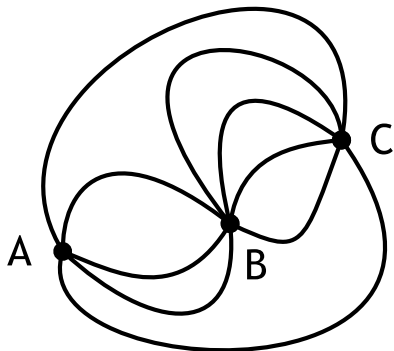
Factorial



# Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city *A* to city *C*, such that each city is visited not more than once.



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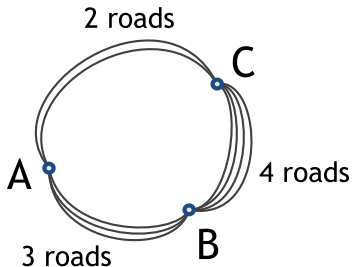
Tree Diagrams

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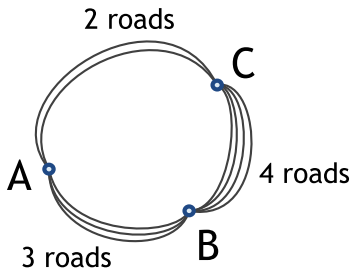
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# Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city *A* to city *C*, such that each city is visited not more than once.



$A \rightarrow C$  or  $A \rightarrow B \rightarrow C$ :

$$2 + 3 \cdot 4 = 14$$

The Rule of Product

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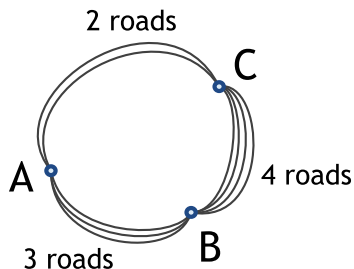
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# Counting round trips

This is a map with three cities, connected by roads.

Count the number of round trips starting in city  $B$ , such that cities  $A$  and  $C$  are visited not more than once.



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# Counting round trips

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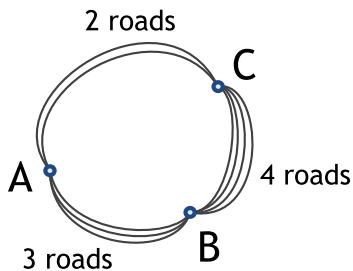
Tree Diagrams

Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city  $B$ , such that cities  $A$  and  $C$  are visited not more than once.

$B \rightarrow A \rightarrow B$	$3 \cdot 3 = 9$
$B \rightarrow C \rightarrow B$	$4 \cdot 4 = 16$
$B \rightarrow A \rightarrow C \rightarrow B$	$3 \cdot 2 \cdot 4 = 24$
$B \rightarrow C \rightarrow A \rightarrow B$	$4 \cdot 2 \cdot 3 = 24$
<hr/>	
Total	73

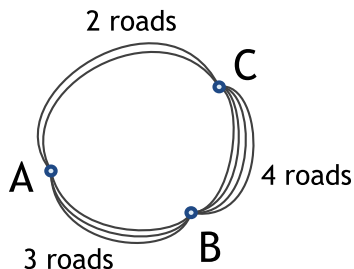


# Counting round trips II

This is a map with three cities, connected by roads.

Count the number of round trips starting in city  $B$ , such that

- 1) cities  $A$  and  $C$  are visited not more than once, and
- 2) each road is used not more than once during a trip.



The Rule of Product

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# Counting round trips II

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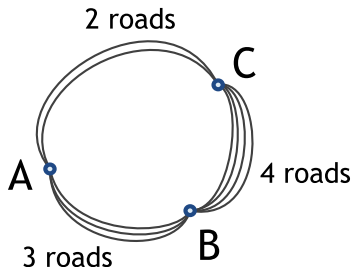
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city  $B$ , such that

- 1) cities  $A$  and  $C$  are visited not more than once, and
- 2) each road is used not more than once during a trip.

$B \rightarrow A \rightarrow B$	$3 \cdot 2 = 6$
$B \rightarrow C \rightarrow B$	$4 \cdot 3 = 12$
$B \rightarrow A \rightarrow C \rightarrow B$	$3 \cdot 2 \cdot 4 = 24$
$B \rightarrow C \rightarrow A \rightarrow B$	$4 \cdot 2 \cdot 3 = 24$
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Total	66



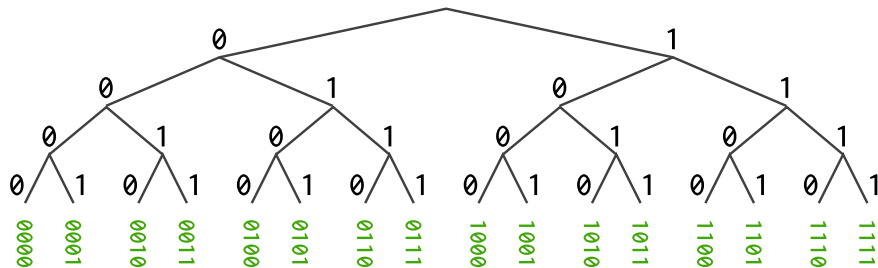
# Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.

Each possible outcome is a leaf of the tree (an endpoint that does not branch).

**Example:** Count all bit strings of length four.



16 strings.

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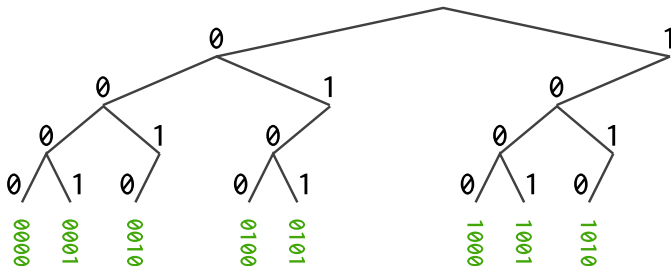
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Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.

Each possible outcome is a leaf of the tree (an endpoint that does not branch).

**Example 2:** Count all bit strings of length four that *do not have two consecutive 1s*.



8 strings.

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

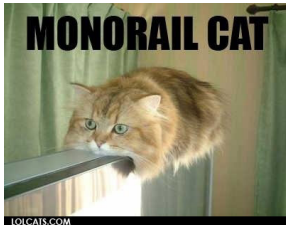
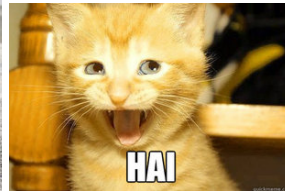
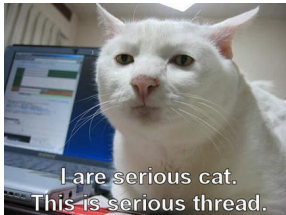
Subtraction Rule

Counting

Tree Diagrams

Factorial

# Ranking cats



The Rule of Product

The Rule of Sum

Finite Sets

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# Ranking cats

In how many different ways can you rank  
a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

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# Ranking cats

In how many different ways can you rank a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

- There are 6 ways to select the first cat,
- 5 ways to select the second cat among the remaining five,
- 4 ways to select the third cat ...
- ...continue the process
- In the end, the only remaining cat takes the last position in the rank.

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$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

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# Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by  $n!$ :

$$1! = 1$$

$$2! = 2 \cdot 1$$

$$3! = 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

and by convention,

$$0! = 1$$

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**Factorial**