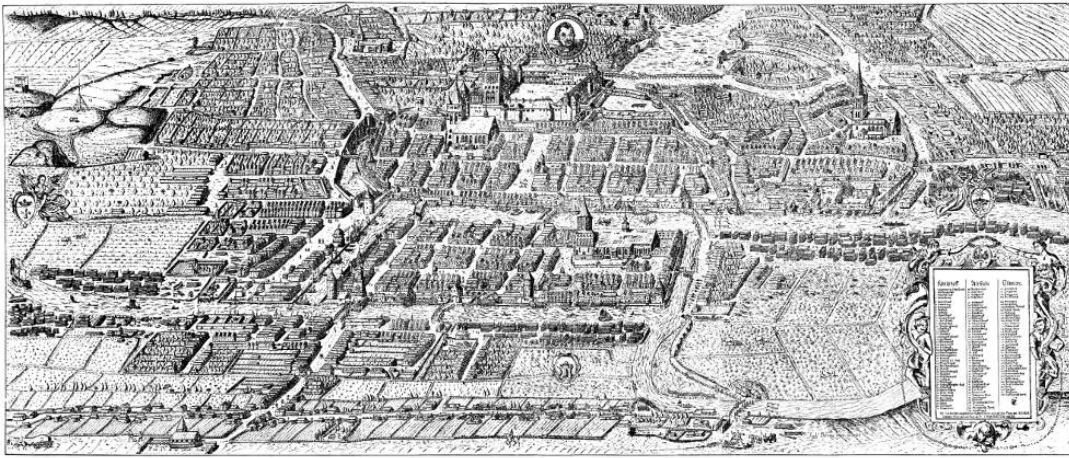
Graphs

City of Königsberg, Prussia, 1735.

Gedenkblatt zur sechshundert jährigen Jubelfeier der Königlichen Haupt und Residenz-Stadt Königsberg in Preußen.



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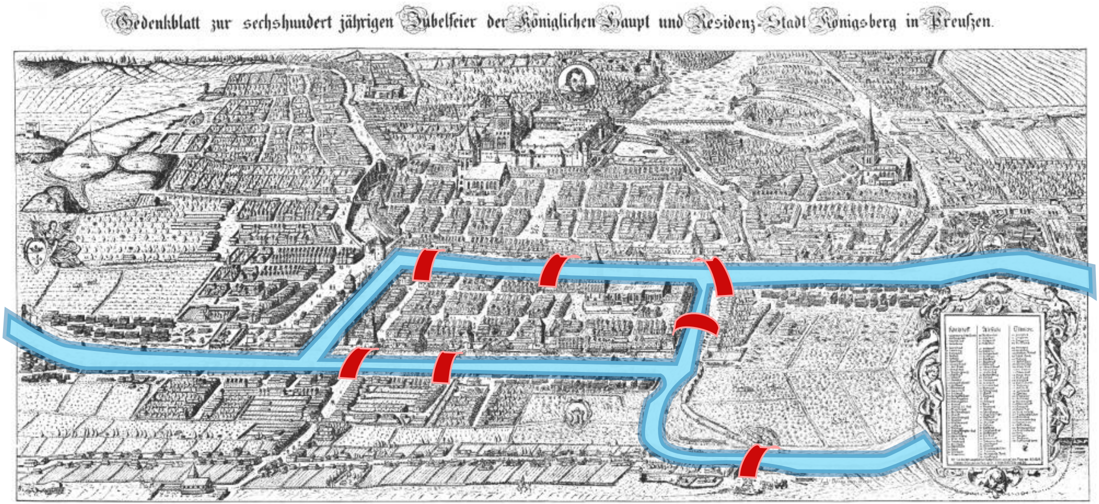

Task: Find a path through the city that would cross each bridge once and only once.


LEONHARD EULER
1707–1783
e-k+f=2
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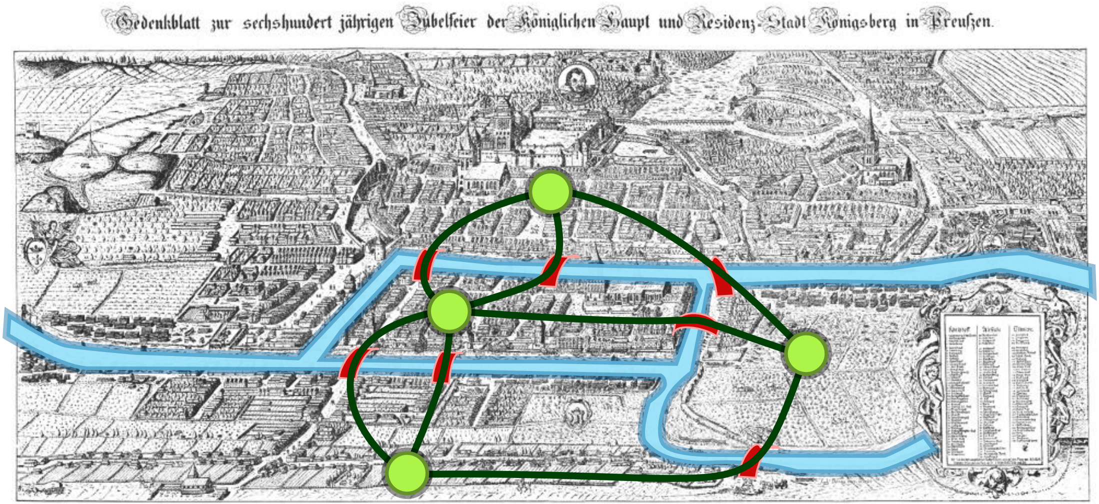
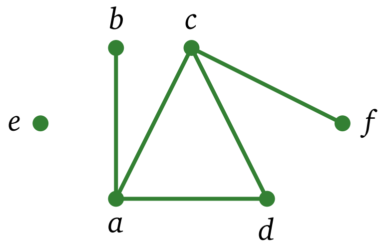
HELVETIA
ANGELO BOOG
2007

City of Königsberg, Prussia, 1735.



Task: Find a path through the city that would cross each bridge once and only once.

Basic definitions



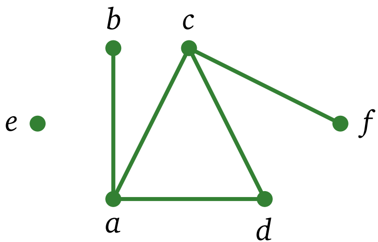
Def. *Graph* $G = (V, E)$ is a set of *vertices* V, with a set of *edges* E between them.

Def. Each edge has *two endpoints*.

Def. An edge *joins* its endpoints, two endpoints are *adjacent* if they are joined by an edge.

Def. An edge is said to be *incident* to the vertices it joins.

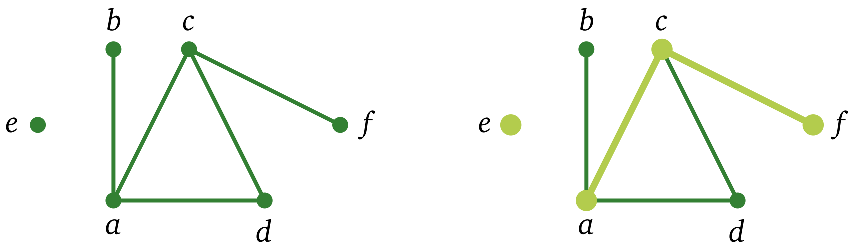
Basic definitions



$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{c, f\}\}$$

Subgraphs



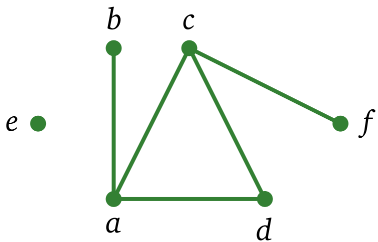
Deleting some vertices or edges from a graph leaves a *subgraph*. Formally:

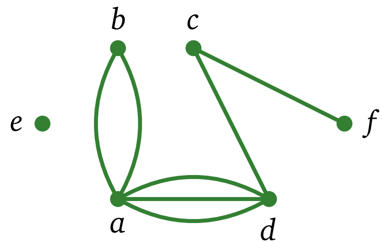
Def. A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ where V' is a nonempty subset of V and E' is a subset of E.

$$V' = \{a, c, f, e\}$$

$$E' = \{\{a, c\}, \{c, f\}\}$$

Variants: Multigraph

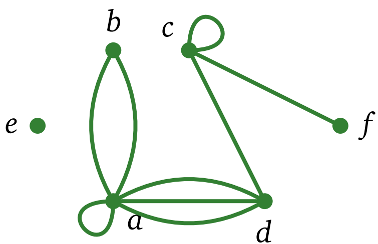




Def. In *simple graphs*, each pair of distinct vertices has at most one edge.

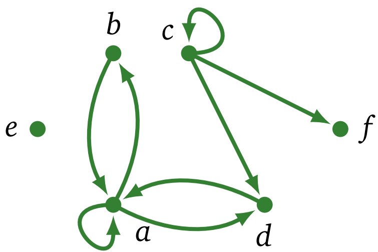
Def. Graphs that may have multiple edges connecting the same vertices are called *multigraphs*

Variants: Graphs with loops



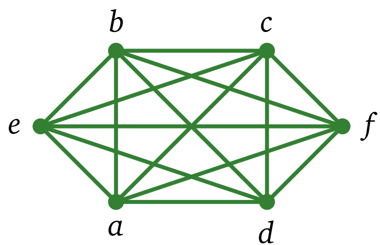
Some graphs that may include *loops*, and possibly multiple edges connecting the same pair of vertices or a vertex to itself.

Directed graphs



Def. In *directed graph* (or digraph) the edges are directed, that is every edge (u, v) is an ordered pair. It starts at u and ends at v.

Complete graph, K_n

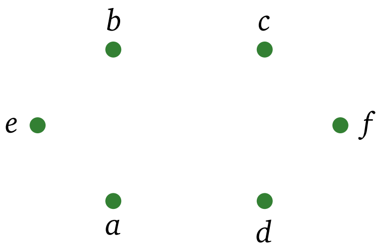


Def. *Complete graph* is a simple graph that has one edge between each pair of vertices.

They are denoted by K_n, where n is the number of vertices.

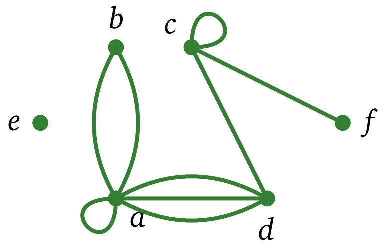
K_6 is in the figure above.

Empty graph



Def. *Empty graph* has empty set of edges.

Degree in undirected graphs

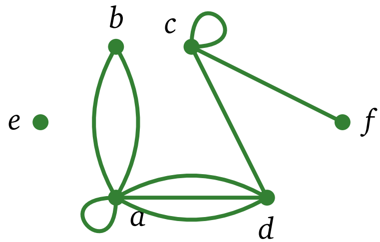


Def. The *degree* of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex v is denoted by $\deg(v)$.

$$\deg(a) = 7, \quad \deg(b) = 2, \quad \deg(c) = 4,$$
$$\deg(d) = 4, \quad \deg(e) = 0, \quad \deg(f) = 1.$$

The handshaking lemma



Lemma (The handshaking lemma)**.** Let (V, E) be an undirected graph with m edges. Then

$$\sum_{v \in V} \deg(v) = 2m.$$

Corollary. An undirected graph has an even number of vertices of odd degree.

Social graphs

1. Prove that there is no group of 7 people such that each person in the group has exactly 3 friends in the group.

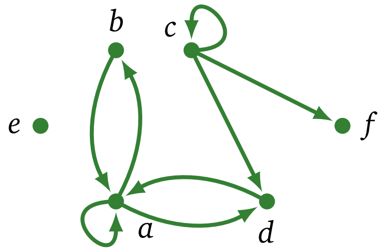
☺☺☺☺☺☺☺

Friendship is always mutual.

That is, in math-speak, the *friendship relationship is symmetric*.

2. Then, try to prove that in any group of $n \geq 2$ people, there are at least 2 people with the same number of friends in the group.

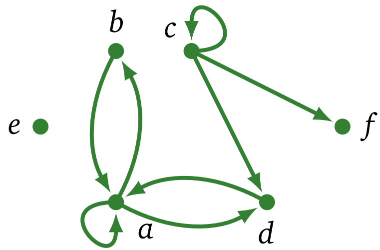
Degree in directed graphs



Def. In directed graphs, there are similar notions of *in-degree* and *out-degree*, denoted by $\deg^-(v)$ and $\deg^+(v)$ respectively

$$\deg^-(a) = 3, \quad \deg^+(a) = 3, \quad \deg^-(b) = 1, \quad \deg^+(b) = 1,$$
$$\deg^-(c) = 1, \quad \deg^+(c) = 3, \quad \deg^-(d) = 2, \quad \deg^+(d) = 1,$$
$$\deg^-(e) = 0, \quad \deg^+(e) = 0, \quad \deg^-(f) = 1, \quad \deg^+(f) = 0.$$

Degree in directed graphs

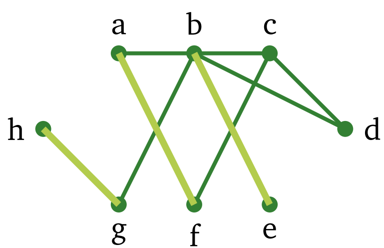


Theorem. Let (V, E) be a directed graph. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|.$$

Matching

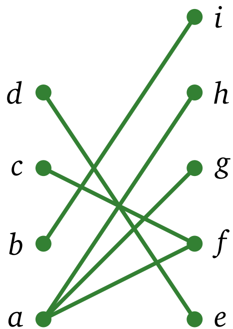
Def. A *matching* M in a simple graph (V, E) is a subset of E such that no two edges from M are incident with the same vertex.



In other words, a matching is a set of *disjoint* edges.

Also, we can introduce maximum, maximal, perfect matchings.

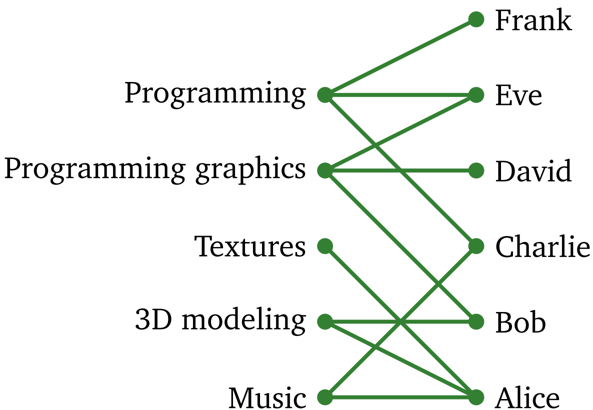
Bipartite graph



Def. A simple graph is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2

Matching

Suppose that there are m employees in a group and n different jobs that need to be done, where $m \geq n$.



Matching

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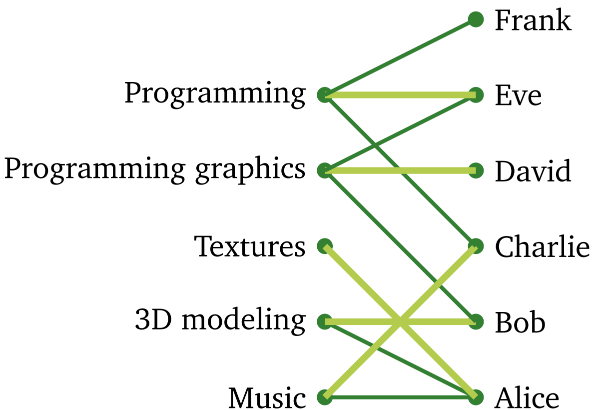
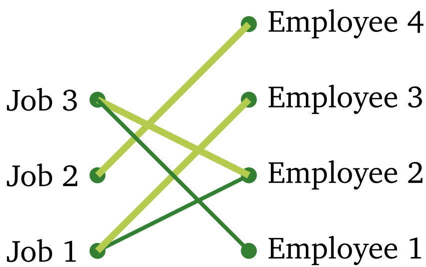


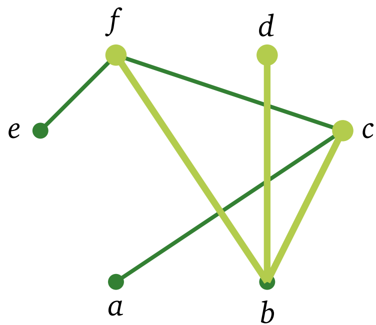
Complete matching from V_1 to V_2

Def. We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a *complete matching from V_1 to V_2* if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.



So, every job is assigned to some employee, and no employee is assigned to more than one job.

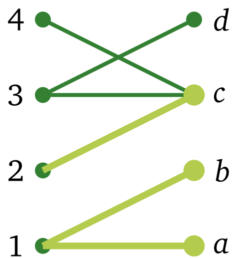
Neighborhood of a vertex



$$N(\{b\}) = \{f, d, c\}$$

Neighborhood of a set of vertices

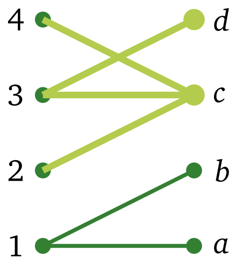
Given a set of vertices S, define $N(S)$ to be the set of all neighbors of S; that is, all vertices that are adjacent to a vertex in S, but not actually in S.



$$N(\{1,2\}) = \{a, b, c\}$$

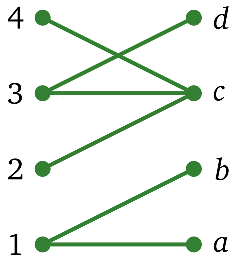
Neighborhood of a set of vertices

Given a set of vertices S, define $N(S)$ to be the set of all neighbors of S; that is, all vertices that are adjacent to a vertex in S, but not actually in S.



$$N(\{2,3,4\}) = \{c, d\}$$

Hall's theorem



Theorem (Hall's Marriage Theorem). The bipartite graph (V, E) with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if

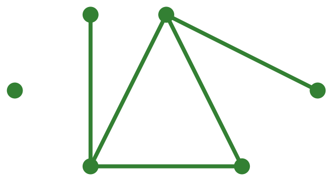
$$|N(A)| \geq |A|$$

for all subsets $A \subseteq V_1$.

Question: Is there a complete matching from $V_1 = \{1, 2, 3, 4\}$ to $V_2 = \{a, b, c, d\}$?

Graph coloring and bipartite graphs

Graph coloring is a task to assign colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

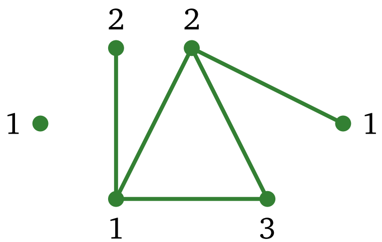


Def. A graph G is *k-colorable* if each vertex can be assigned one of k colors so that adjacent vertices get different colors.

Theorem. A simple graph is *bipartite* if and only if it is *2-colorable*.

Graph coloring and bipartite graphs

Graph coloring is a task to assign colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.



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Theorem. A simple graph is *bipartite* if and only if it is *2-colorable*.

Graph coloring

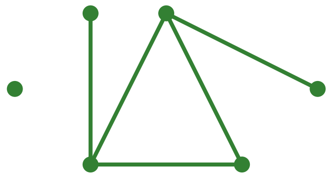
Def. The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. It's denoted by $\chi(G)$.

The following theorem helps to estimate the chromatic number.

Theorem. A graph G with maximum degree at most k is $(k+1)$-colorable:

$$\max_{v \in V}(\deg(v)) \leq k \quad \rightarrow \quad \text{G is } (k+1)\text{-colorable.}$$

Graph coloring



$$\max_{v \in V}(\deg(v)) \leq k \quad \rightarrow \quad \text{G is } (k+1)\text{-colorable.}$$

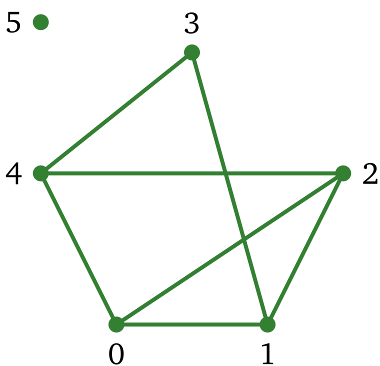
Proof. The theorem can be proved by induction.

The base case. A graph with $|V| = 1$ does not have edges, so the maximum degree is 0, and the graph is 1-colorable.

Inductive step. Assume that a graph with $n-1$ vertices and maximum degree at most k is $(k+1)$ colorable.

Now, prove that a graph with n vertices and maximum degree at most k is $(k+1)$ colorable ...

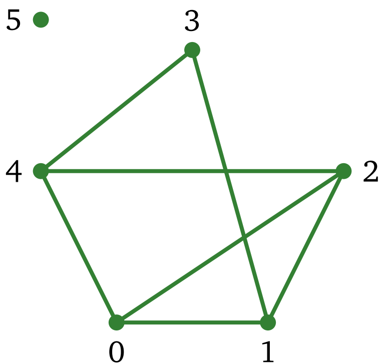
Representing graphs



n vertices and m edges.

How to represent a graph in a computer program?

Representing graphs



n vertices and m edges.

Adjacency Matrix

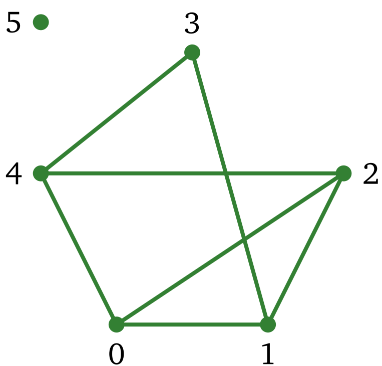
2-D array $n \times n$.

$a[i, j] = 1$ if there is an edge between i and j.

	0	1	2	3	4	5
0		1	1		1	
1	1		1	1		
2	1	1			1	
3		1			1	
4	1		1	1		
5						

Takes $O(n^2)$ space.

Representing graphs



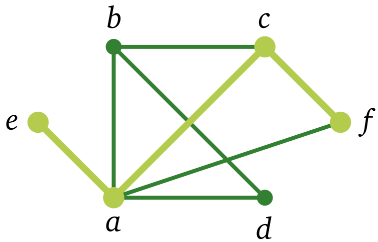
n vertices and m edges.

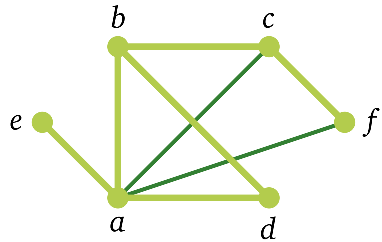
Adjacency List

$$\mathrm{adj}(0) = [1, 2, 4]$$
$$\mathrm{adj}(1) = [0, 2, 3]$$
$$\mathrm{adj}(2) = [0, 1, 4]$$
$$\mathrm{adj}(3) = [1, 4]$$
$$\mathrm{adj}(4) = [0, 2, 3]$$
$$\mathrm{adj}(5) = []$$

Takes $O(nm)$ space.

Path





Def. A *path* from s to t is a sequence of edges

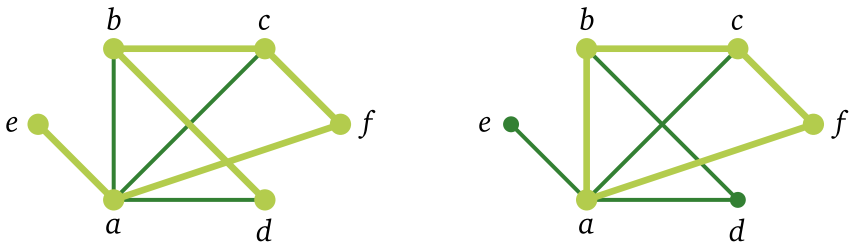
$$\{x_0, x_1\},\ \{x_1, x_2\},\ \ldots\ \{x_{n-1}, x_n\},$$

where $x_0 = s$, and $x_n = t$.

Def. The *length* of a path is the number of edges in it.

$$\{e, a\}\ \{a, b\}\ \{b, d\}\ \{d, a\}\ \{a, b\}\ \{b, c\}\ \{c, f\}$$

Simple path. Cycle



Def. A *simple path* is a path that does not contain the same edge more than once.

Def. A path is called a *cycle* (or *circuit*) if its first and last vertices are the same, and its length is greater than 0.

Def. A *simple sycle* is a cycle that does not contain the same edge more than once.