Relations. Partial orders.

Relations

Remember that a relation is a subset of the Cartesian Product of two sets.

For example,

$$R = \{(a,b) \in A \times B \mid \text{some property holds}\}$$

$$R \subseteq A \times B$$

For convenience, we adopt the following infix notation:

$$\text{when} \quad (a,b) \in R, \quad \text{we write} \quad aRb$$

Relations. Infix notation

It is originated from the relations like $=$, $\leq$, $\geq$, $<$, and $>$.

$$(1,2) \in R_{(<)} \quad \text{we usually write} \quad 1 < 2$$

$$(3,3) \in R_{(=)} \quad \text{we usually write} \quad 3 = 3$$

Divisibility is a relation on $\mathbb{N}$ too. And we use infix notation:

$$(15,60) \in R_{(divides)} \quad \text{we write} \quad 15 \mid 60$$

Relations on the same set

What if the sets A and B are the same?

$$R \subseteq A \times A$$

For example, $=$, $\leq$, $\geq$, $<$, $>$ are relations on $\mathbb{N}$. That is, these relations are subsets of $\mathbb{N} \times \mathbb{N}$.

Def. A relation on the set A is

- *reflexive* if $\forall x \in A : xRx$.
- *symmetric* if $\forall x, y \in A : xRy \rightarrow yRx$.
- *antisymmetric* if $\forall x, y \in A : (xRy \wedge yRx) \rightarrow x = y$.
- *transitive* if $\forall x, y, z \in A : (xRy \wedge yRz) \rightarrow xRz$.

Relations on the same set

- *reflexive* if $\forall x \in A : xRx$.
- *symmetric* if $\forall x, y \in A : xRy \to yRx$.
- *antisymmetric* if $\forall x, y \in A : (xRy \wedge yRx) \to x = y$.
- *transitive* if $\forall x, y, z \in A : (xRy \wedge yRz) \to xRz$.

	reflexive?	symmetric?	antisymmetric?	transitive?
$x \equiv y \pmod 5$	Yes	Yes	No	Yes
$x \mid y$	Yes	No	Yes	Yes
$x \leq y$	Yes	No	Yes	Yes

Partial orders

Def. A relation is a *partial order* if it is reflexive, antisymmetric, and transitive.

An example, the “divides” relation on the natural numbers is a partial order:

- It is reflexive because $x \mid x$.
- It is antisymmetric because $x \mid y$ and $y \mid x$ implies $x = y$.
- It is transitive because $x \mid y$ and $y \mid z$ implies $x \mid z$.

The $\leq$ relation on the natural numbers is also a partial order. However, the $<$ relation is not a partial order, because it is not reflexive; no number is less than itelf.

Partial orders

Often a partial order relation is denoted with the symbol

$$\preceq$$

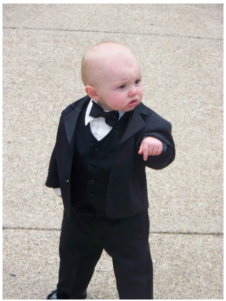
instead of a letter, like R.

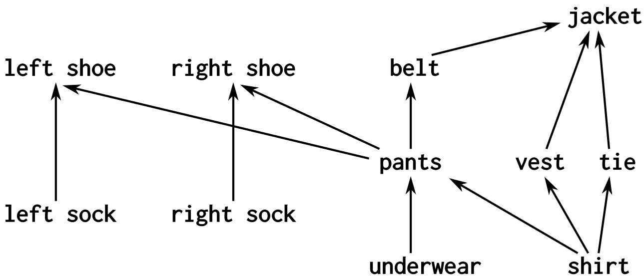
This makes sense since the symbol calls to mind $\leq$, which is one of the most common partial orders.

$x \preceq y$ it reads as "x precedes y".

Partially ordered sets

Def. If $\preceq$ is a partial order on the set A, then the pair $(A, \preceq)$ is called a *partially-ordered set* or *poset*.

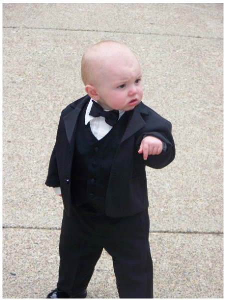


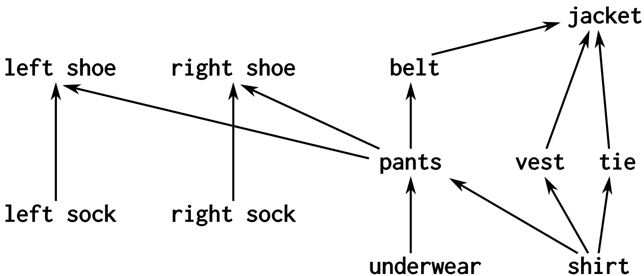


Def. The elements x and y of a poset $(A, \preceq)$ are called *comparable* if either $x \preceq y$ or $x \preceq y$.

When x and y are elements of A such that neither $x \preceq y$ nor $y \preceq x$, x and y are called *incomparable*.

Hasse diagram





This graph is called the *Hasse diagram* for the poset $(A, \preceq)$.

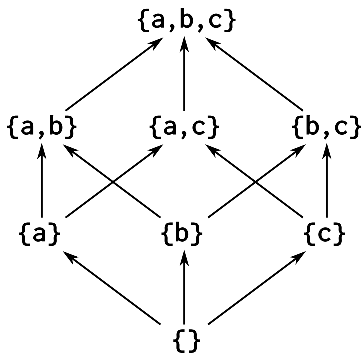
For a and b from A, we draw an edge from a to b if $a \preceq b$.

Self-loops and edges implied by transitivity are omitted.

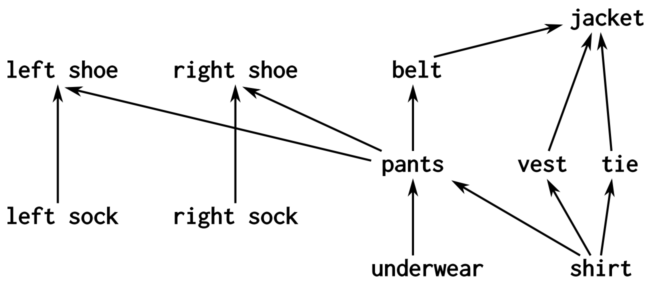
Hasse diagram

Consider a poset $(\mathscr{P}(A), \subseteq)$ for $A = \{a, b, c\}$.

Its Hasse diagram:



Minimal and maximal elements



In a poset $(A, \preceq)$, an element $x \in A$ is *minimal* if there is no other element $y \in A$ such that $y \preceq x$.

Similarly, an element $x \in A$ is *maximal* if there is no other element $y \in A$ such that $x \preceq y$.

There are four minimal elements.

Partially ordered sets

Theorem. A poset $(A, \preceq)$ has no directed cycles other than self-loops, that is, there is no sequence of $n \geq 2$ distinct elements $a_i \in A$ such that

$$a_1 \preceq a_2 \preceq a_3 \preceq a_4 \preceq \ldots \preceq a_{n-1} \preceq a_n \preceq a_1$$

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Proof. Suppose that for some $n \geq 2$ such sequence $a_1 \ldots a_n$ exists.

Recall that the partial order is a transitive, antisymmetric, and refelxive relation.

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Recall that the partial order is a transitive, antisymmetric, and refelxive relation.

Since it's transitive: $a_1 \preceq a_2$ and $a_2 \preceq a_3$, therefore $a_1 \preceq a_3$.

Similarly, we prove that $a_1 \preceq a_4$, $a_1 \preceq a_5$, $\ldots$, $a_1 \preceq a_n$.

Thus $a_1 \preceq a_n$ and $a_n \preceq a_1$.

But $\preceq$ is antisymmetric, and therefore $a_1 = a_n$. This contradicts the supposition that $a_1, \ldots a_n$ are $n \geq 2$ distinct elements! Thus there is no such directed cycle.

Total order

Def. A *total order* is a partial order in which every pair of elements is comparable.

$(A, \preceq)$ is a total order if for every $x, y \in A$, either $x \preceq y$ or $y \preceq x$.

The $\leq$ relation on natural numbers is a total order. However, the "divides" relation on the same set $\mathbb{N}$ is not.

Question: Given a parially ordered set $(A, \preceq)$, can we make a total order $\preceq_T$ that is "compatible" with the given partial order $\preceq$? (Compatible in the sense that the total order never violates the given partial order)

Topological sort

Def. A *topological sort* of a poset $(A, \preceq)$ is a total order $\preceq_T$ s.t.

$$x \preceq y \quad \text{implies} \quad x \preceq_T y.$$

Theorem. Every finite poset has a topological sort.

Lemma. Every finite poset has a minimal element.