Relations. Functions. Bijection and counting.

Cartesian products

Given two sets

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 3, 4\}$$

Their Cartesian product

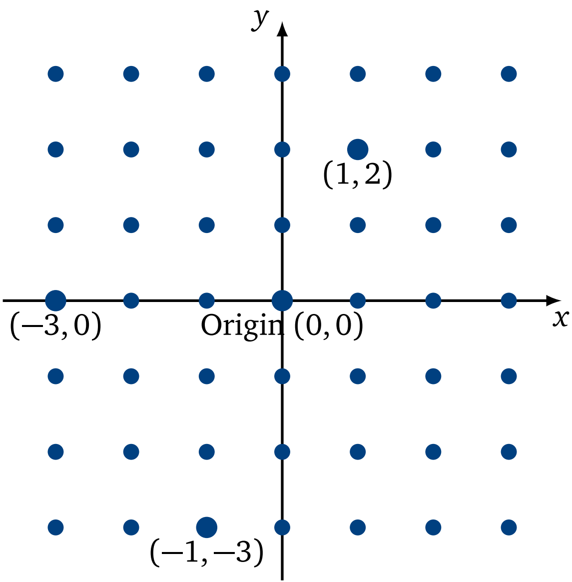
$$\begin{aligned} A \times B = \{&(1,1),\ (2,1),\ (3,1),\\ &(1,2),\ (2,2),\ (3,2),\\ &(1,3),\ (2,3),\ (3,3),\\ &(1,4),\ (2,4),\ (3,4)\} \end{aligned}$$

Question: What is the cartesian product of $\mathbb{Z} \times \mathbb{Z}$?

($\mathbb{Z}$ is the set of all integers)

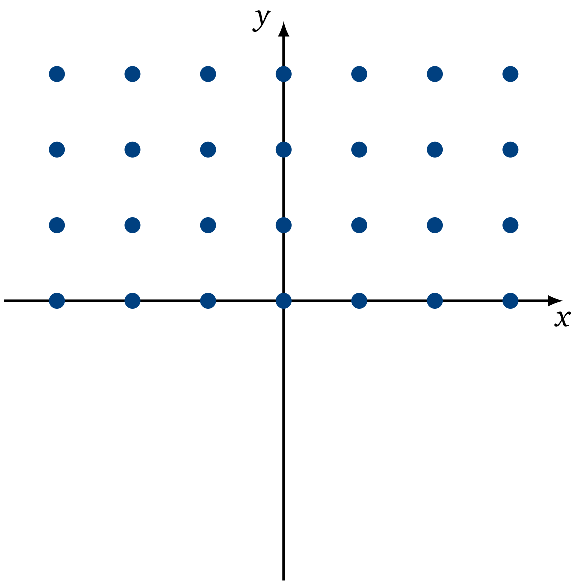
Cartesian product $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$

Cartesian product
Functions
Bijection
Inclusion-Exclusion

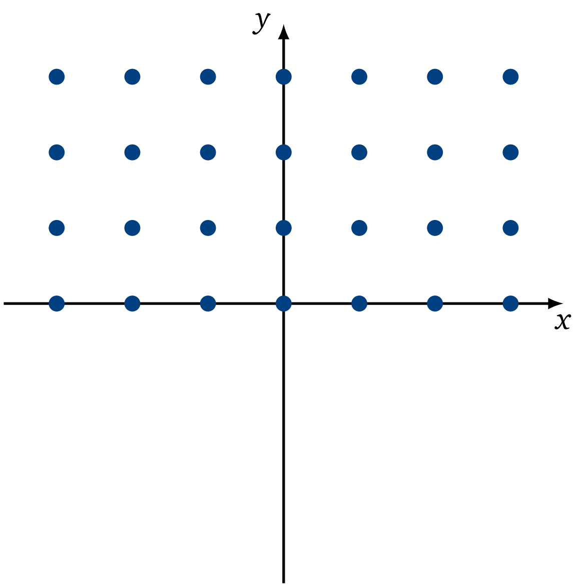


All pairs of integers are in $\mathbb{Z}^2$, for exmaple $(1,2) \in \mathbb{Z}^2$

Is it a Cartesian product?



Cartesian product of $\mathbb{Z} \times \mathbb{N}$



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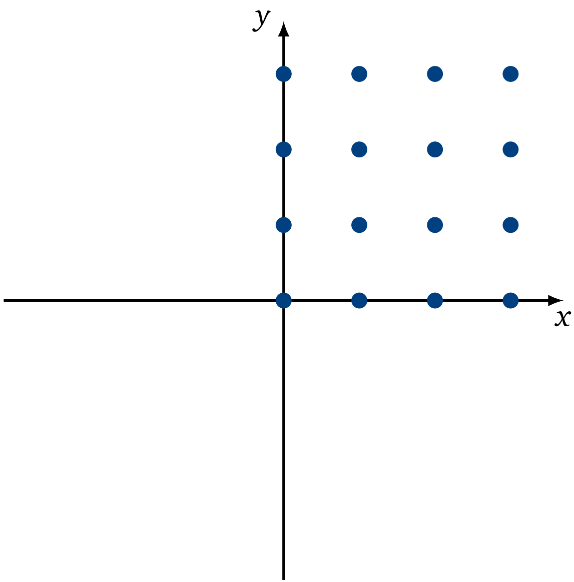
Is it a Cartesian product?



Cartesian product

Functions

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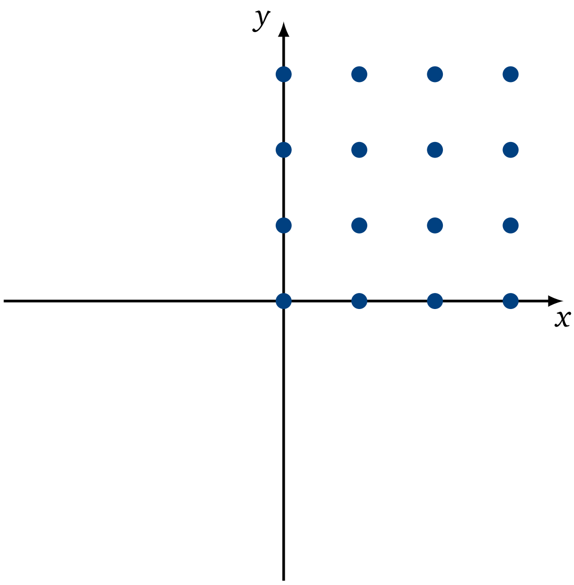
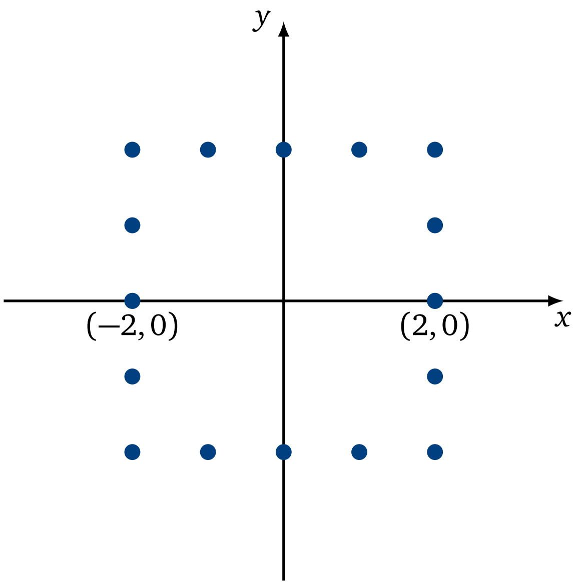
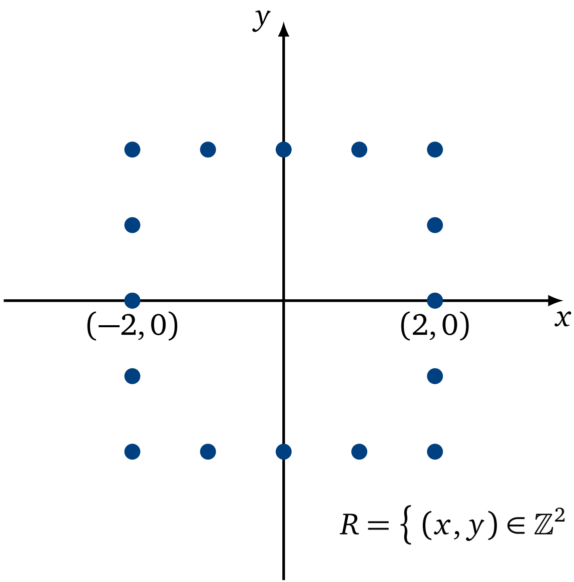
Cartesian product of $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$



Is it a Cartesian product?



Is it a Cartesian product? No



$$R = \{ (x, y) \in \mathbb{Z}^2 \mid \max(|x|, |y|) = 2 \} \subseteq \mathbb{Z}^2$$

Is it a Cartesian product? No

Cartesian product
Functions
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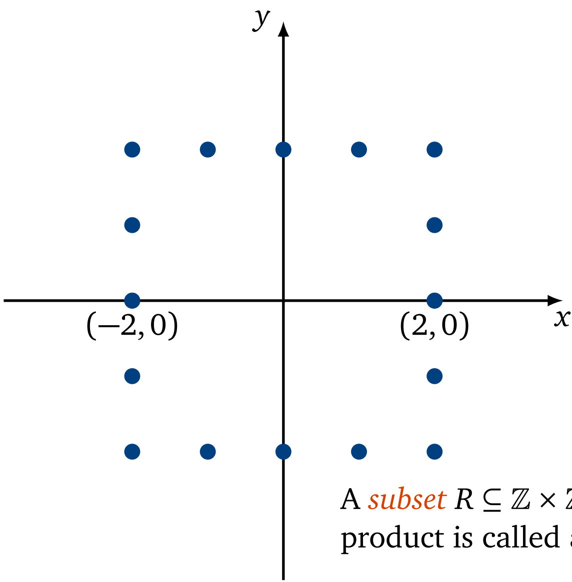

A *subset* $R \subseteq \mathbb{Z} \times \mathbb{Z}$ of a Cartesian product is called a *relation*.

Relations

Cartesian product
Functions
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Inclusion-Exclusion

Given any two sets

$$A = \{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\} \qquad \text{and} \qquad B = \{1, 2, 3, \ldots\}$$

Def. A *subset R of the Cartesian product $A \times B$* is called a *relation* from the set A to the set B.

$$R = \Big\{ (\spadesuit, 99), (\heartsuit, 15), (\clubsuit, 10^5), (\clubsuit, 1), (\clubsuit, 15) \Big\} \subseteq A \times B$$

Relations

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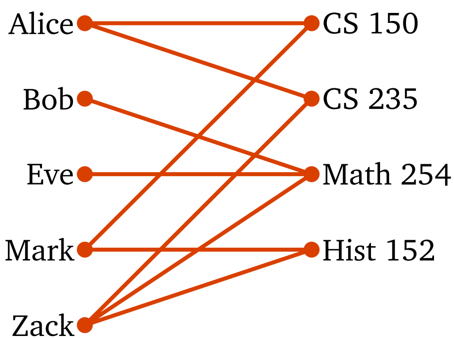
Example of a relation:

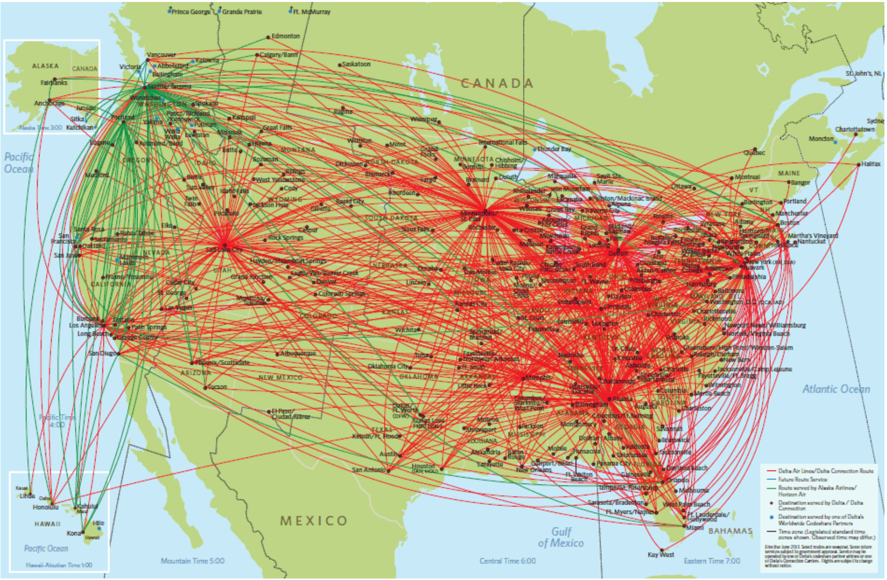
$$S = \text{set of students}$$

$$C = \text{set of classes}$$

$$R = \{(s, c) \mid \text{student } s \text{ takes class } c\}$$

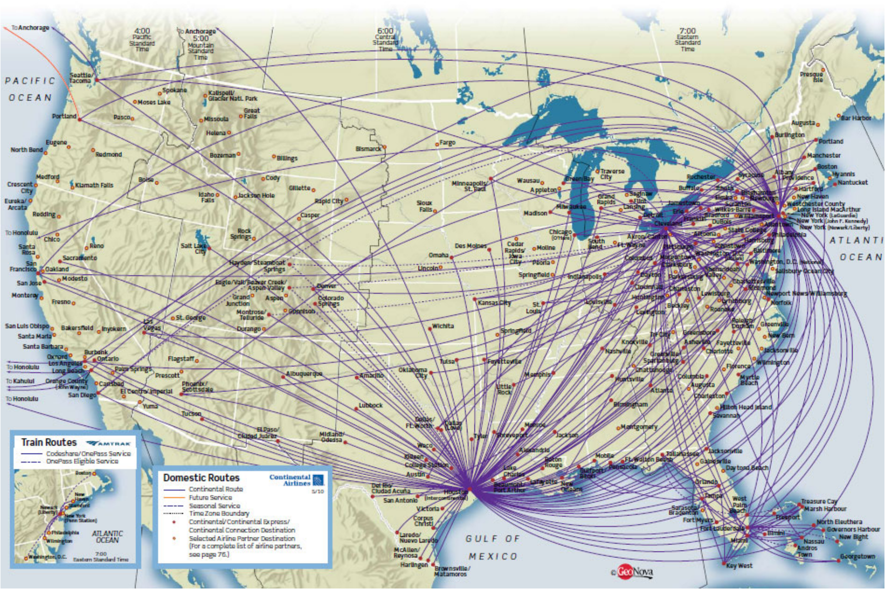


Airline route map



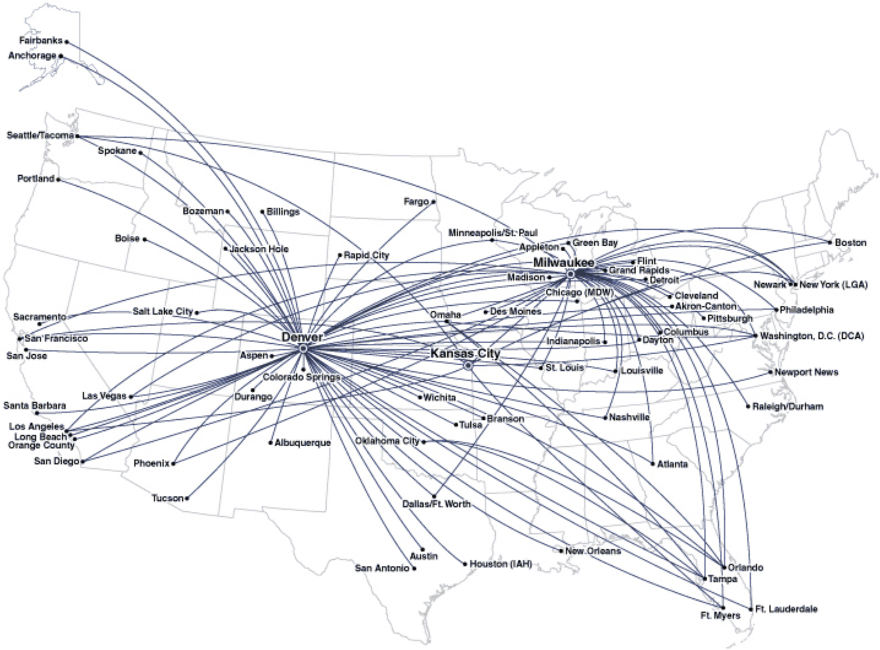
We connect two cities if the airline operates a flight from one to the other. Is it a relation?

Airline route map



$$Routes \subseteq Cities \times Cities$$

Airline route map



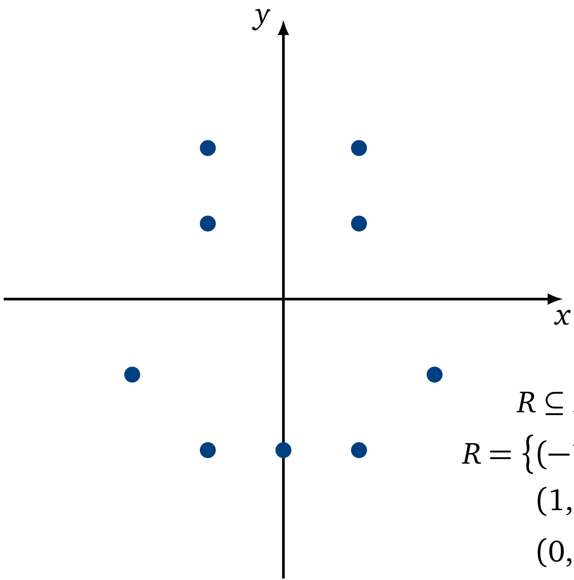
Cartesian product

Functions

Bijection

Inclusion-Exclusion

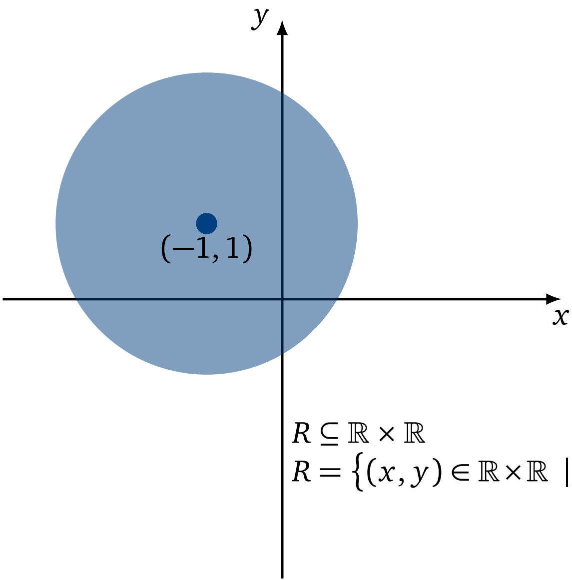
Any subset of $A \times B$ is a relation



$$R \subseteq \mathbb{Z} \times \mathbb{Z}$$

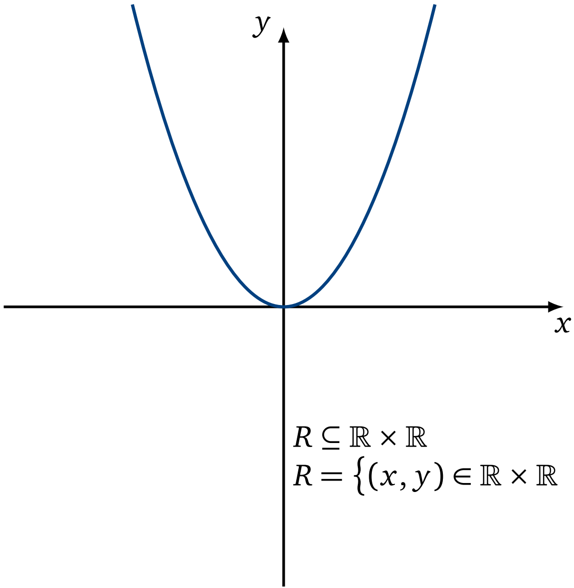
$$R = \{(-1,2),(1,2),(-1,1),\\ (1,1),(-2,-1),(-1,-2),\\ (0,-2),(1,-2),(2,-1)\}$$

Any subset of $A \times B$ is a relation



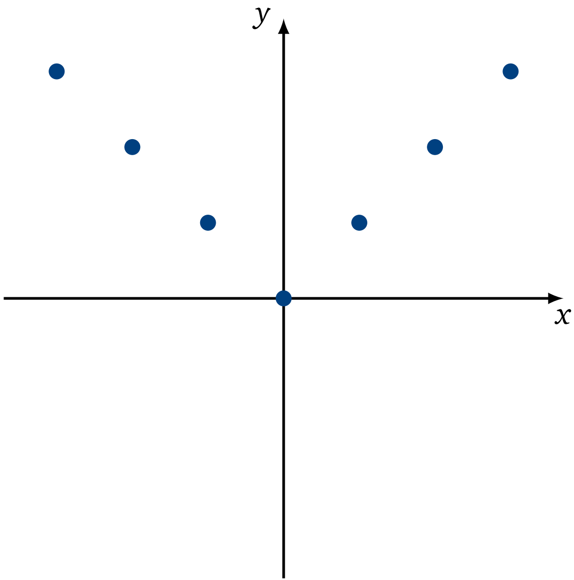
$$R \subseteq \mathbb{R} \times \mathbb{R}$$
$$R = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x+1)^2 + (y-1)^2 \leq 2\right\}$$

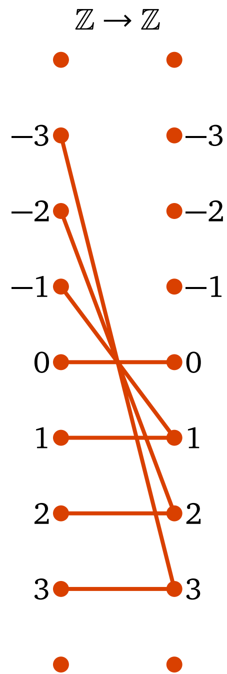
A function is a relation too!



$$R \subseteq \mathbb{R} \times \mathbb{R}$$
$$R = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \underline{y = f(x)}\right\}$$

Relation $\{(x, y) \mid y = |x|\}$





Cartesian product

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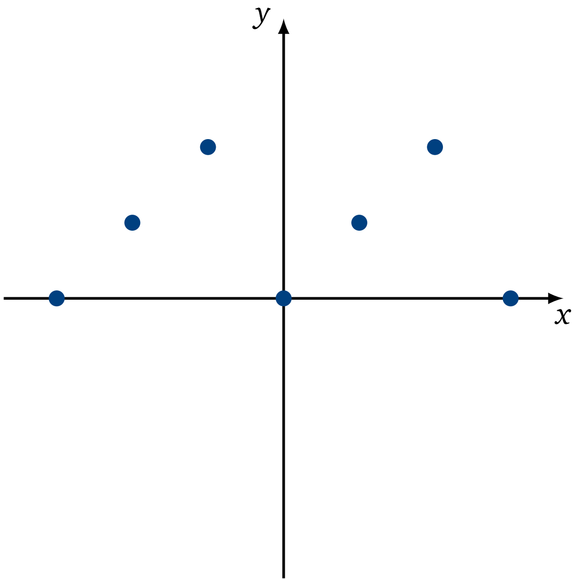
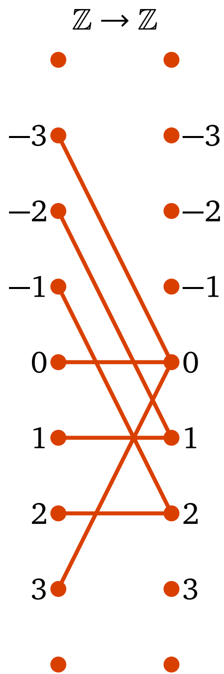
Relation $\{(x, y) \mid y = x \ \mathbf{rem}\ 3\}$





Cartesian product

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Functions

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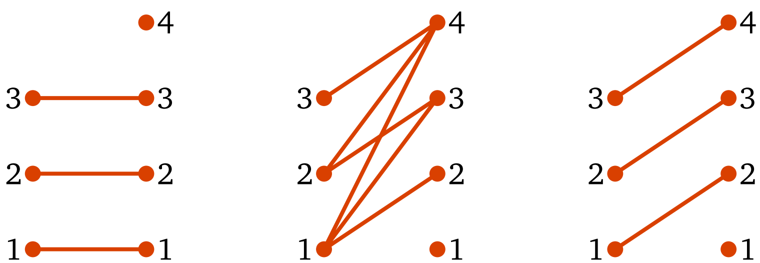
Def. A relation $R \subseteq A \times B$ is a *function* (a functional relation) if for every $a \in A$, there is at most one $b \in B$ so that $(a, b) \in R$.

$$A = \{1, 2, 3\}, \quad B = \{1, 2, 3, 4\}$$

$R_1 = \{(1,1),\ (2,2),\ (3,3)\}$

$R_2 = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$ $\leftarrow$ Not a function

$R_3 = \{(1,2),\ (2,3),\ (3,4)\}$



Functions

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Functional relation $R \subseteq A \times B$ defines a unique way to map each element from the set A to an element from the set B.

There is a well-known and convenient notation for functions:

$$f(a) = b \qquad \text{where } a \in A \text{ and } b \in B$$

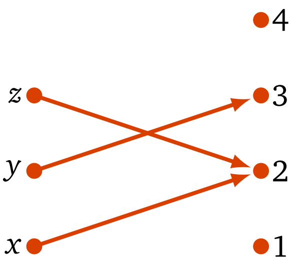
It maps elements from A to B:

$$f : A \longrightarrow B$$

$$A \xrightarrow{f} B$$

Functions

Def. For the function $f : A \to B$, set A is called *domain*, and set B is called *codomain*.



$$f : A \to B$$

$$\text{domain}(f) = A = \{x, y, z\}$$
$$\text{codomain}(f) = B = \{1, 2, 3, 4\}$$
$$\text{image}(f) = f(A) = \{2, 3\}$$

Def. $f(a)$ is the *image of a point* $a \in A$.

Def. The *image of a function* f, denoted by $f(A)$, is the set of all images of all points $a \in A$

$$f(A) = \{x \mid \exists a \in A\,(f(a) = x)\}.$$

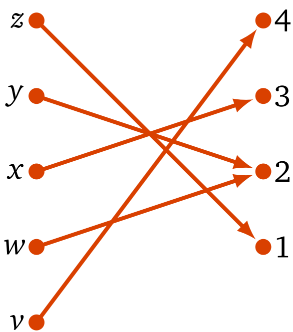
The image of a function is also called *range*.

Onto

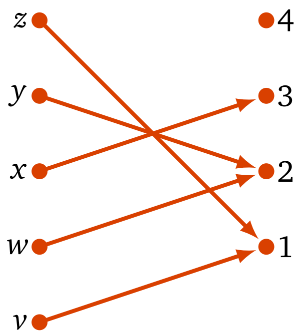
Def. A function $f : A \to B$ is called *onto* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, the image $f(A)$ is the whole codomain B.

$f : A \to B$ is onto



$g : A \to B$ is not onto



One-to-one

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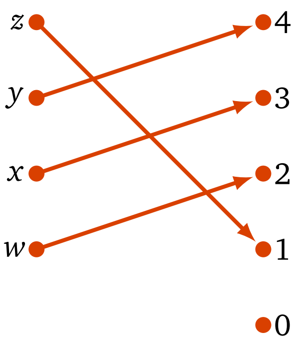
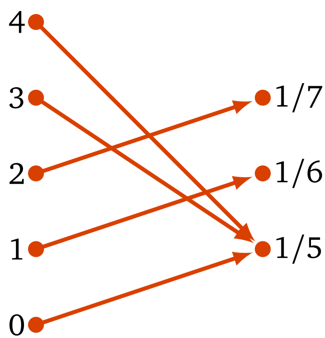
Def. A function $f : A \to B$ is said to be *one-to-one* if and only if $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.

$f : A \to B$ is one-to-one



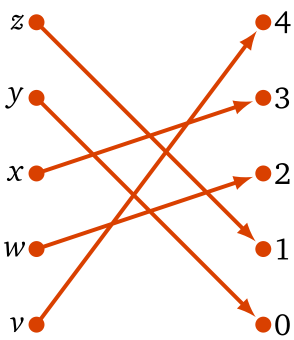
$g : B \to C$ is not one-to-one



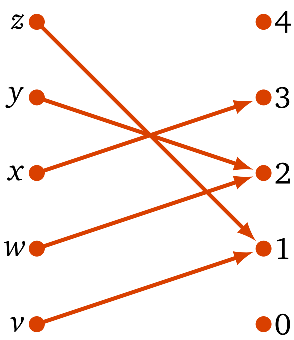
Bijection

Def. The function f is a *bijection* (also called one-to-one correspondence) if and only if it is both one-to-one and onto.

$f : A \to B$ is a bijection



$g : A \to B$ is not a bijection



Bijection

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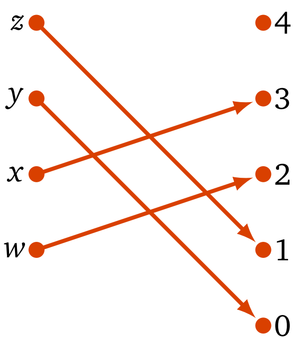
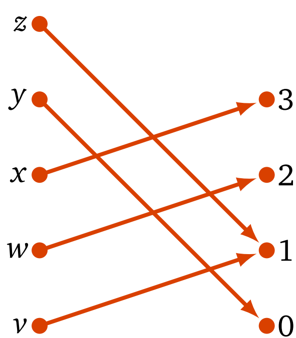
$f : A \to B$ is one-to-one, but not onto



$g : C \to D$ is onto, but not one-to-one



So, both functions are not bijections.

Bijection. Observation

Cartesian product

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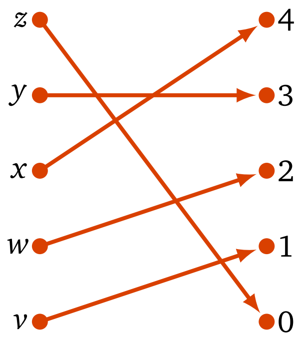
Bijection Rule. Given two sets A and B, if there exists a bijection

$$f : A \to B, \quad \text{then } |A| = |B|.$$



We can count the size of the set A, instead of the size of B!

Bijection Rule.

Consider two similar problems:

(a) How many bit strings contain exactly three 1s and two 0s?

$$11010$$

(b) How many strings can be composed of three 'A's and five 'b's so that an 'A' is always followed by a 'b'?

$$AbAb\,bAb\,b$$

We show that this two problems are equivalent by constructing a bijection.

Bijection Rule.

Let X be the set of bit strings

$$X = \{11010, \dots\}$$

and Y be the set of 'A' and 'b' strings

$$Y = \{AbAbbAbb, \dots\}$$

We can construct a bijection $f : X \to Y$:

1 gets replaced by Ab
0 gets replaced by b

Bijection Rule.

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$$
\begin{aligned}
f: 11100 &\mapsto Ab\ Ab\ Ab\ b\ b \\
11010 &\mapsto Ab\ Ab\ b\ Ab\ b \\
11001 &\mapsto Ab\ Ab\ b\ b\ Ab \\
10110 &\mapsto Ab\ b\ Ab\ Ab\ b \\
10101 &\mapsto Ab\ b\ Ab\ b\ Ab \\
10011 &\mapsto Ab\ b\ b\ Ab\ Ab \\
01110 &\mapsto b\ Ab\ Ab\ Ab\ b \\
01101 &\mapsto b\ Ab\ Ab\ b\ Ab \\
01011 &\mapsto b\ Ab\ b\ Ab\ Ab \\
00111 &\mapsto b\ b\ Ab\ Ab\ Ab
\end{aligned}
$$

Function f is one-to-one and onto, so it is a bijection. Therefore, the cardinalities of two sets are equal: $|X| = |Y| = \binom{5}{3} = 10$.

Bijection. Observation

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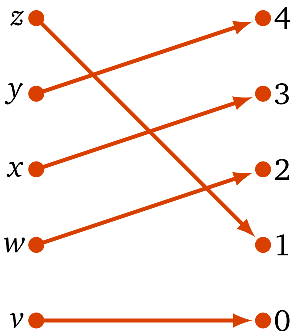
Observation For every bijection $f : A \longrightarrow B$, exists an *inverse* function

$$f^{-1} : B \longrightarrow A$$

$f : A \longrightarrow B$



Bijection. Observation

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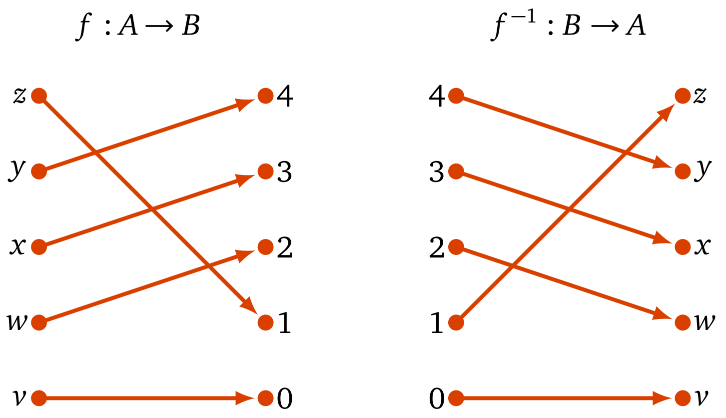
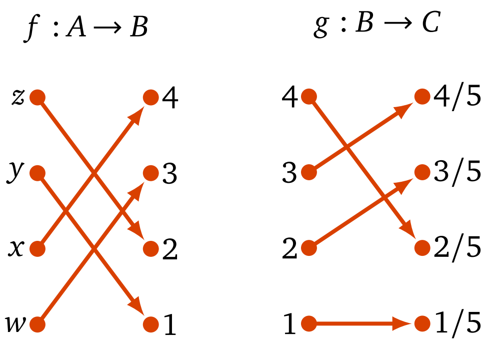


The inverse function is a bijection too.

Bijection. Observation

Given two bijections $f : A \to B$ and $g : B \to C$.
Consider their composition

$$h(x) = g(f(x))$$



Bijection. Observation

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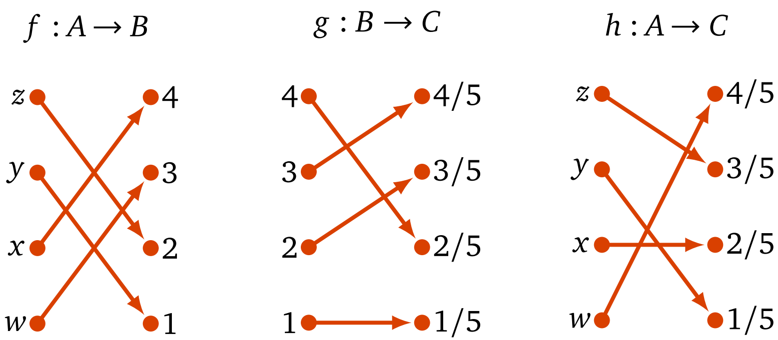

$h : A \to C$ is a bijection, and therefore $|A| = |C|$.

Bijection. Counting subsets

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Bijection. Counting subsets

Cartesian product

Functions

Bijection

Inclusion-Exclusion

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Let's find a bijection f between the power set

$$\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, \ldots\}$$

and the set of bit stings of length 5:

$$\{0, 1\}^5 = \{00000, 00001, 00010, 00011, \ldots\}$$

Bijection. Counting subsets

$$f : \mathscr{P}(\{a,b,c,d,e\}) \to \{0,1\}^5$$

0s and 1s encode the membership of the five elements of $\{a,b,c,d,e\}$

$$\begin{aligned} f : \varnothing &\mapsto 00000 \\ \{a\} &\mapsto 10000 \\ \{b\} &\mapsto 01000 \\ \{a,b\} &\mapsto 11000 \\ \{c\} &\mapsto 00100 \\ \{a,c\} &\mapsto 10100 \\ \{b,c\} &\mapsto 01100 \\ \{a,b,c\} &\mapsto 11100 \\ &\ldots\text{skipping um.. 23 subsets} \\ \{a,b,c,d,e\} &\mapsto 11111 \end{aligned}$$

The cardinality

$$\left|\{0,1\}^5\right| = 2^5 = 32$$

Therefore, by the bijection rule,

$$\left|\mathscr{P}(A)\right| = 2^5$$

Inclusion-Exclusion principle

Cartesian product
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Inclusion-Exclusion

We remember the subtraction rule for the union of two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can it be generalized for a union of n sets

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - \langle \textit{something} \rangle ?$$

Inclusion-Exclusion principle

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Can it be generalized for a union of n sets

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - \langle something \rangle ?$$

Of course, it can!

Inclusion-Exclusion principle

Cartesian product
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Union of three sets

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3| = \quad & |A_1| + |A_2| + |A_3| \\
& - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| \\
& + |A_1 \cap A_2 \cap A_3|
\end{aligned}
$$

$$|\{1,2,3\} \cup \{2,3,4\} \cup \{3,4,1\}| = 3+3+3-2-2-2+1 = 4$$

Inclusion-Exclusion principle

Cartesian product

Functions

Bijection

Inclusion-Exclusion

Union of n sets

$\lvert A_1 \cup \ldots \cup A_n \rvert =$	the sum of the sizes of the individual sets
minus	the sizes of all two-way intersections
plus	the sizes of all three-way intersections
minus	the sizes of all four-way intersections
plus	the sizes of all five-way intersections
	etc.